Mathematics for Policy and Planning Science

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Abstract

Introduction to Mathematics for Policy and Planning Science (1st half): the course, the instructor, and the field. Introduction to mathematical modeling.

The Times They Are A-Changin'

Come gather 'round people, where ever you roam
And admit that the waters around you have grown
And accept it that soon, you'll be drenched to the bone
If your time to you is worth saving
Then you'd better start swimming or you'll sink like a stone
For the times, they are a-changin'.

- Lost decades, Chinese economic growth, Lehmann shock, Brexit, Trump: the times are changing.

We Need to Change with the Times

- It seems like everybody in Japan hopes for a return to the age of *monozukuri* and lifetime employment. Many professors in *Shakō* teach about *machizukuri*, which is pretty close. Unfortunately, these days Japan has a lot more jobs pouring drinks than pouring concrete, even with earthquake disasters and the 2020 Tokyo Olympics.
- Those are still value-added areas in goods production where Japan can lead —but they probably won't lead to great increases in employment.
- To employ all her people at high wages, Japan needs to become a service economy. And not just services, but one focused on innovation in customized products—just as the U.S. has already done.
- China is not far behind Japan, Europe, and Korea on the same trajectory.
- We'll talk about the COVID-19 pandemic later.

The road to tomorrow

- Many of you will prefer “traditional” paths to employment, with big manufacturing companies or the public sector. But I hope that many will turn to ventures and the service sector. Modern economies need leaders in innovation. And many say government and traditional mass production industries should innovate, too.

- To lead in innovation at the “global” level, one must engage in *service engineering*.
  - Use the web to enable (or for information services, to implement) customized products.
  - Theory-based implementation and marketing (not experimentation and *kaizen*) to take advantage of “market windows”. *E.g.*, fast fashion (success), 3-CD boomboxes (failure), soft drinks (mixed).
  - Precise calculations to achieve larger margins in mature industries.

- “Engineering” is theory-based and quantitive, yet requires human expertise and judgment, and experimentation.

Modeling

First, we need to talk about *models*.

- A model is a *structure* for thinking about something complex.
- Plastic model airplanes and model cars emphasize the shape, often including the shape of parts you normally can't see, such as the turbines in a jet engine and the hydraulic lines that control the brakes under a car.
- Fashion models also emphasize shape, but they do so dynamically. Professional models don't just stand there, they walk around and move their limbs. Here what you see is much closer to what you get, except that fashion models are a lot skinnier and usually taller than everyday humans.
- But both plastic models and fashion models are *incomplete* (the engines don't work) and *idealized* to some extent (the ladies in my family don't look like models on TV, but they wish they did). We say they are *abstract*.

Modeling, *cont.*

- Models guide our behavior, often prescriptively (or *normatively*). That's why we talk about "model students." Teachers want other students to *emulate* those models.
- Models may be *informative*, as in artist's models. An artist's model is often not sitting for a portrait. The purpose of the model is so that the artist can support the *art* with a certain amount of realism.
- A map (graphic image) is a model.
- In mathematical logic, a model is a map (function). In all of our examples, there are maps: the shape of a plastic model to the real airplane, the shape of the human fashion model's body to the ideal, and to the clothes they wear, the behavior that teachers want to the behavior of a particular student called "a model student," from a human girl to a Picasso painting.

There was a model ...

Removed due to copyright concerns. Please search Google:

`https://www.google.com/search?q=picasso+woman+images`

Mathematical modeling

- Consider a 3D printer (or any printer—an image is also a model of the thing it depicts). How does it work? You feed it numbers to tell it where to put tiny drops of plastic (3D) or ink (2D).
- The image produced by the printer is a physical representation (model) of the object. The list of numbers the computer sends to the printer is a quantitative (mathematical) representation of the object.
- Of course, once we have a quantitative representation, we can abstract further by using *functions* to generate the numbers.
- Once we have a functional representation, we can use *algebra*, *calculus*, and even more advanced mathematics to analyze our model (because, of course, the function is another representation, or model, of whatever we are studying).

Causal modeling

- In science, and even in daily life, the most important models are *causal models*. They explain behavior we observe in terms of causes and effects.
- If we ask an attractive person to go out with us, and they say no, we *ask* them *why*. That's the most direct way to acquire a model of their behavior. Or, if they say yes, we may *imagine* they find us attractive too. Imagining is a very dangerous way to acquire models. In the case of the dating example, that's how marriage frauds make money!
  - Of course, we have no choice but to *start* with our imaginations. But we should not *rely* on such models until we have *verified* them.
- We ask *why* when we're doing science, too. Why does a person become sick with the *disease* COVID-19? Our model is that the *necessary* condition is infection with the *virus* SARS-CoV-19. Both cause and effect are actually in the name: "SARS" stands for "severe acute respiratory syndrome" which tells us about the disease, "CoV" stands for "corona virus," and "19" for the year of discovery, 2019. Showing cause and effect is why scientists like names like "SARS-CoV-19" although ordinary people just say "the coronavirus."

Modeling COVID-19

Let's think about an example of building a model.

- What do we know about disease, for example, the flu? It's pretty much the same every year nowadays: in the autumn we go to the doctor, get vaccinated, a few people get sick, a very few get very sick and die, and then we repeat the following year.
- This is the *very* simplest model: a *constant*. What happens the next time is what happened the last time.
- COVID-19 is new (that's the "novel" in "novel corona virus"). By that very fact, the idea of a constant model (next year will be the same is this year) is undermined. Next time (tomorrow) is not going to be the same as last time (yesterday). We see that in the papers every day: new cases—and new deaths.
- Yet the constant model plays a fundamental part in the political economy of the pandemic.

A bad model

- The political argument is that we see that the business shutdown orders had a big economic effect:
  - before the order, most people took a few precautions and then went to school, work, and play, but
  - after the order, many business shut, people stayed home, and
  - we see a very large negative impact on the economy.
- Therefore, it was a bad idea to shut all the businesses before we were sure that the virus was spreading explosively.
- This is based on the *constant model*. Why? Because the standard of comparison for "very large negative effect" is *last period's GDP* (or *employment*, *etc.*).
- The implied assumption is "if we had no shutdown order, the economy would work the same as last year," *i.e.*, the constant model.

Rejecting the bad model

- OK, it's the constant model. Is that *bad*? After all, it works for the flu, and business activity was going on as before.
- The problem is that we know from the experience of Wuhan (China), Bergamo (Italy), and New York (US) that COVID-19 is different from the flu. We *don't* have *any* vaccine, a *large fraction* of the population gets sick, and *many* people get sick enough to die.
- Finally, it had a great effect on the economy even before shutdown orders in those places, even if you *only* count economic losses due to sick workers and shortages from falling production, and the like.
- The constant model is *untenable* (we can't defend it).
- To estimate the costs (economic and otherwise) of business shutdowns, we need to evaluate a *counterfactual*: "What would the level of economic activity be *with* COVID-19 but *without* the shutdown?

Counterfactuals

- How do we evaluate a counterfactual? You guessed it, I'm sure. We build a model. *We have no choice but to build a model.*
- Sometimes we can avoid explicit modeling. If we're lucky, we have appropriate data and we use analogical reasoning: *this* situation is like *that* situation, so the outcome *this* time should be like the outcome *that* time.
  - You may notice that this is just an alternative description of the constant model.
- Building a model of the effect of COVID-19 on a national economy is hard. So hard that I don't know of any professionals willing to publish theirs yet.
- Part of economic modeling must be modeling the disease's *medical effects*, which is not something we can do in Shako, especially not this class.
- But there's a component of that model that we can at least get started on: the *epidemiology* of the virus (*i.e.*, the scientific study of how it spreads).

The simplest usable epidemiological model

We need to keep asking "why?" and what that means for our model.

- Why can't we use the constant model? We know from the worst-hit cities that the SARS-CoVID-19 virus has a *doubling time*, which varies from place to place (and responds to policy) and is estimated at 2–7 days.
- Assuming we take no special action, and a doubling time of 2 days (worst case), starting from *one* infected person, we get the following table:

day	0	2	4	6	8	10	12	14	16	18	20	22	24
count	1	2	4	8	16	32	64	128	256	512	1,024	2,048	4,096

day	26	28	30	32	34	36	38
count	8,192	16,384	32,768	65,536	131,072	262,144	524,288

day	40	42	44	46	48	50
count	1,048,576	2,097,152	4,194,304	8,388,608	16,777,216	33,554,432

Table 1: Exponential growth model

Simplest epidemiological model, *cont.*

- Continuing the process of Table 1:

day	52	54	56	58	60
count	67,108,864	134,217,728	268,435,456	536,870,912	1,073,741,824

day	62	64	66	68
count	2,147,483,648	4,294,967,296	8,589,934,592	17,179,869,184

Table 2: Exponential growth model, *cont.*

- In other words, if I'm sick today, on May 9 all of you have gotten sick, on May 25 the whole university is infected, on June 2 all of Tsukuba, on June 17 all of Kanto, on June 20 all of Japan, on June 28 all of East Asia, on July 2 the whole world, on July 4 ... uh, wait ...() oops.
- **Lesson #1:** You can run, but you can't hide from exponential growth.
- **Lesson #2:** Exponential growth is a bad model.

Shifting gears to define a better model

- As we'll see later, the exponential growth model is not a bad model in the way that the constant model is. The reason the constant model is *just plain bad* is that without data, it predicts anything you can imagine.
  - I can imagine 10 million people in Tokyo with viral pneumonia from COVID and 10,000 ventilators—two weeks later Tokyo would literally be a ghost town.
  - Politicians seem to imagine 5000 people sick with COVID, and plenty of ventilators for people with asthma (like me) too.

  Arguing about our imaginings is useless.

- But to see that exponential growth is a somewhat useful model it's helpful to use different mathematics, namely *continuous time*. Our calculations were done with *discrete time*, calculating only for every two days. What about odd days? Can we say something about hours or minutes?

Discrete time to continuous time

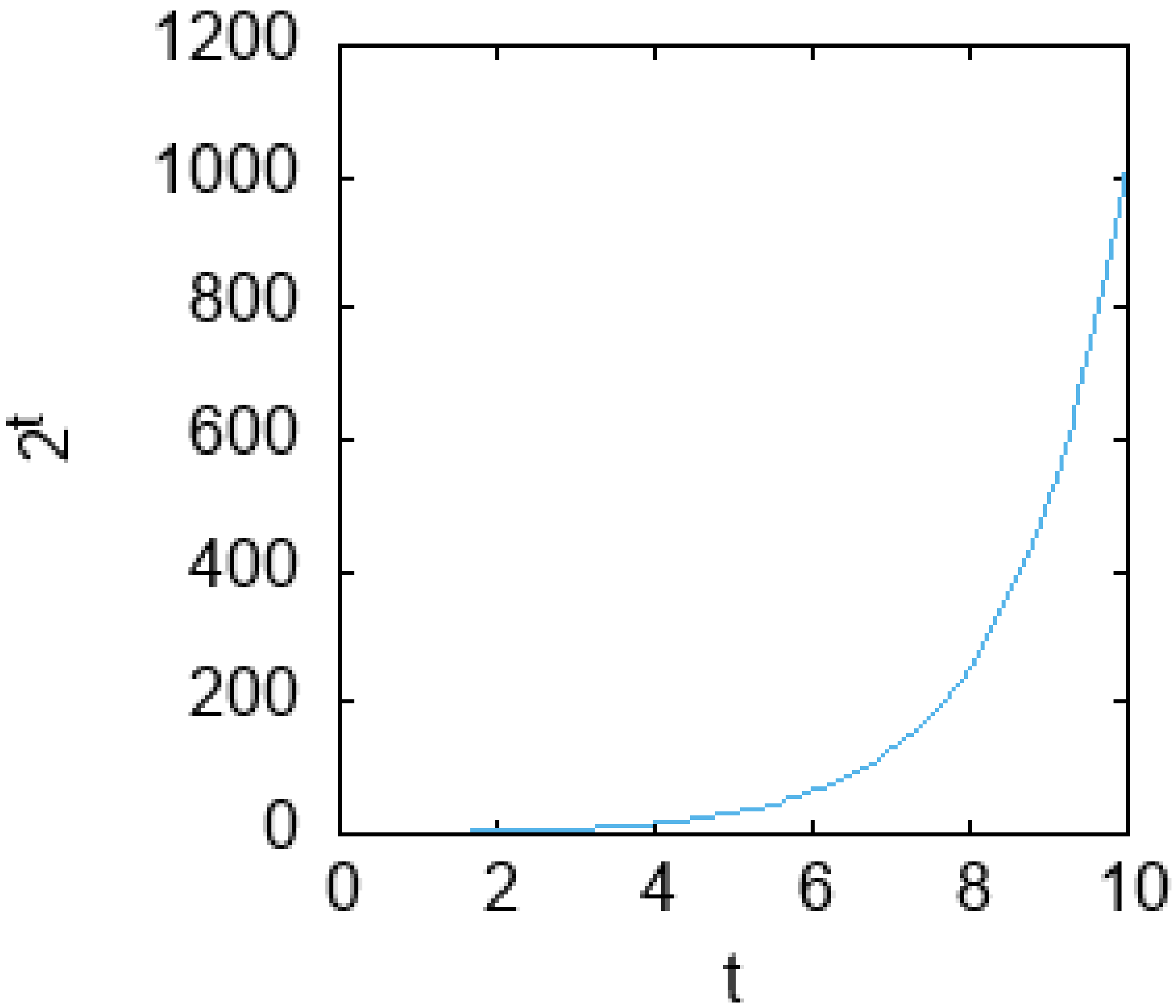
- A bit of thought will show that after t days, we have $2 \times 2 \times \cdots \times 2$ (multiply by 2, $\frac{t}{2}$ times), or $N = 2^{\frac{t}{2}}$ infected individuals. So if we allow t to be odd numbers, or fractional numbers, we can still calculate this (with a computer).
- But this is not very helpful. The first change we're going to make in our model is to change the doubling time so that we can't have more sick people than we have people!
- To express this, the trick is to look at the relationship between the *number of infected individuals* and the *rate of increase of infected individuals*.
  - On any day we can count the infected individuals N.
  - On the next day we'll have $2N$.
  - The *rate of increase* is $2N - N = N$.

  We have expressed the rate of increase directly in terms of N.
- In continuous time, we use calculus and write $\frac{dN}{dt} = N$.
- If we integrate, we get *the* exponential function $N(t) = \exp(t) = e^t$.

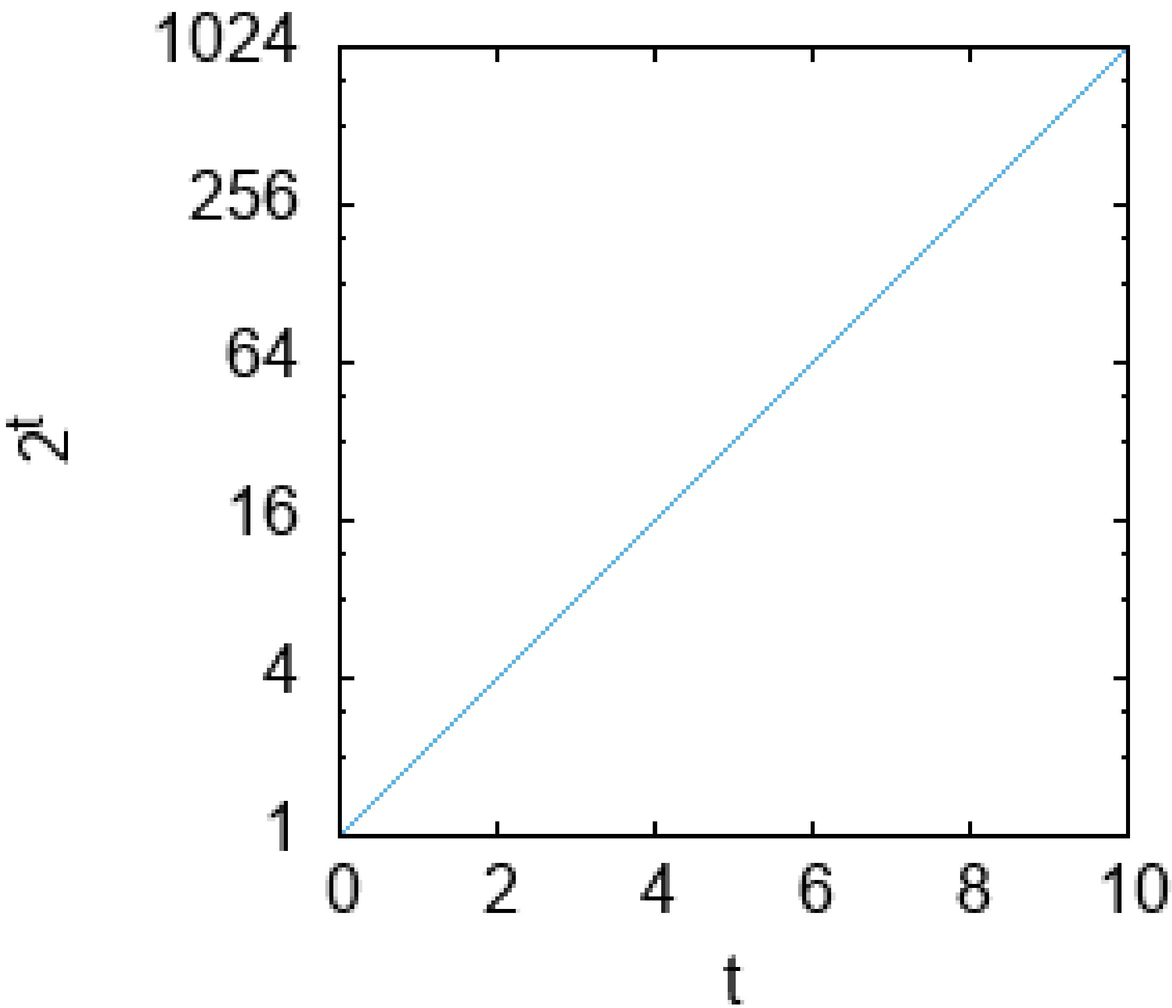
Exponential functions

- For our purpose, *the* exponential function e^t is *defined* by the *differential equation* $\frac{dN}{dt} = N$.
- e is an irrational number, approximately 2.718281828459045 according to my iPhone.
- There are many exponential functions, such as $2^{\frac{t}{2}}$ (and 2^t), but e^t is *the* exponential function because the differential equation is so simple.
- In fact, all exponential functions can be expressed as $f(t) = Ae^{\alpha t}$, and they all have linear differential equations $\frac{df}{dt} = \alpha f(t)$.
- Although the *rate of increase* $\frac{df}{dt}$ changes over time, the *rate of growth* $\frac{df/dt}{f} = \alpha$ does not.
- Try the exponential function $2^{\frac{t}{2}} = e^{0.346573590279973\,t}$ with even numbers t!

Graph(s) of exponential growth



Exponentials, logarithms, and graphs

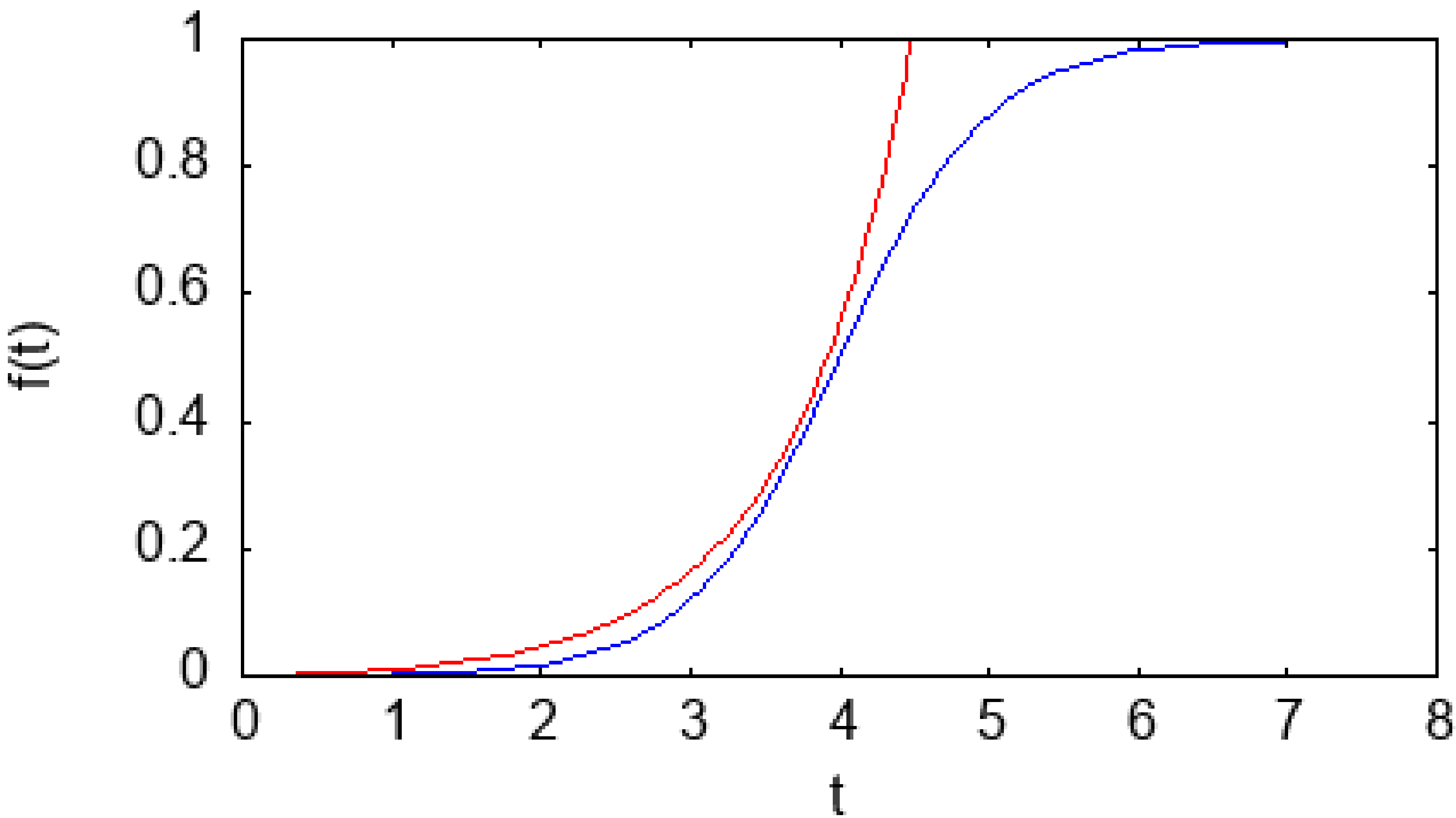


Logistic growth

- How is it that exponential growth crashes through the upper limit of population? One explanation is that it "doesn't know" that only individuals who *aren't infected* can *change to* infected.
- One way to generate a mathematical model is to consider α as the *hazard rate*: the chance that an infected person will infect someone they run into. Since there are $N(t)$ infected individuals meeting other people, the rate at which infections increase is $\frac{dN}{dt} = \alpha N$.
- But already infected individuals don't "get" infected, so the chance that an infected individual meets an uninfected individual is $\frac{\bar{N}-N}{\bar{N}}$, where $\bar{N}$ is the total population (the bar on the top symbolizes "maximum infections").
- Now the hazard rate for infections is $\alpha\frac{\bar{N}-N}{\bar{N}} = \alpha(1 - \beta N)$, where $\beta = \frac{1}{\bar{N}}$, and the differential equation is $\frac{dN}{dt} = \alpha N(1 - \beta N)$.
- This modification would be very tedious in discrete time.

The solution to logistic growth

Compare logistic growth $f(t) = \frac{e^t}{e^t + e^{-t}}$ (blue) with exponential growth (red).
(The exponential growth curve is rescaled to nearly match the logistic curve.)



I am still not happy with logistic growth

- *Why* do we care so much about COVID-19? Because some infected individuals get sick and *die*, a minor model change ($\bar{N}$ changes over time).
- That's the worst case, but infected individuals usually *recover*.
  - When recovered, they may be *susceptible* (they can get infected again), decreasing N, **or**
  - When recovered, they may be *immune* (they cannot get infected again), decreasing N where it means "infectious individuals" (the second factor in the right hand side of the differential equation), but not N where it means "nonsusceptible individuals" (the N in the third factor).

  Either would change the model a lot. Both make the mathematics harder.
- *Why* do people get infected (and then sick)? Because they meet others. But it's not uniform, expressible by a single hazard rate. In fact, the probability of meeting depends on *both* individuals in an encounter. This requires a *social network model*, which generally can't be solved by algebra or calculus, a very big change in the model.

Interpreting logistic growth

- Can we interpret logistic growth realistically for a disease?
- Yes! But it's not a very intuitive way to think about it.
- In the discussion of hazard rate, we need to think about *why* people are removed from the susceptible population.
- One logically correct reason is that with a new virus, it makes sense to talk about the people who never had it as "susceptible" and those who have ever had it as "not susceptible".
- This is not a good model of a *disease*, but it's a logically acceptable way to generate the logistic model.
  - Here, my standard for "good model" is "useful for public health policy."

Picturing the SIR model

- The logistic model is about the best we can do with a single equation model that has a strong theoretical foundation, even though it's not very helpful in predicting hospital demand.
- The SIR model divides the population into three subpopulations (SIR = Susceptible, Infected, Recovered/Immune). Only the Infected population requires treatment, and only the Infected population can infect others.
  - We ignore (*abstract from*) the fact that there are degrees of sickness from "asymptomatic" to "fatal".
- Then the equations are

$$
\begin{aligned}
\frac{dN}{dt}(t) &= \alpha I(t)(1 - \frac{N(t)}{\bar{N}}) \\
R(t) &= N(t-14) \\
I(t) &= N(t) - R(t) \\
S(t) &= \bar{N} - N(t)
\end{aligned}
$$

Still not good enough

- The SIR model is good enough to give a picture that looks a lot like the graphs used to demonstrate the “flatten the curve” effect of social distancing.
- Can’t we stop there?
- Unfortunately not. We know that for individual and economic reasons, amount of social distancing varies greatly among individuals. We must look at a social network model to *estimate* the flattening effect we can get from a policy (more counterfactuals!)
- We also have experience with multiple “waves” of a disease, both very regular ones (the annual “flu season”), and single episodes (the 1918 influenza pandemic, where the second wave caused the most deaths).
- In fact, the 1918 pandemic teaches us that different policies give different curves, and quite different second and third waves. We need models to predict those too.
- The presence of immunity is not known yet, nor is it binary. Some diseases admit nearly complete and permanent immunity (measles), while others aren’t complete (many cold viruses), and others temporary (influenza).

Mathematics: The good, bad, and scary

- The *good* part of mathematics is that *you* don't have to do (much) mathematics. The hardest parts are done by some genius (often a couple centuries ago), and much of the routine part can be done by software (*Mathematica*, *Maxima*, *GAMS*), which will do it right (if you tell it the right thing to do!)

- The *bad* part of mathematics is that there's *lot* of it, you have to choose appropriate mathematics and set it up correctly, and there's always somebody around who knows more mathematics than you (mostly your advisor).

- The *scary* part of mathematics is that there's no way to get around the need for modeling, and the software *can't* do modeling for you, and your advisor *won't* (or at least *shouldn't*) do modeling for you.

What is mathematics? In plain English

- When studying mathematics, you learn many calculations. But this can't be very important; computers calculate better and faster than you do, and you will use them in advanced study and at work.
- By studying mathematics, you learn to create *formal representations of models* which can be calculated. This is probably the most important thing to take away from this class: the idea that mathematics *is not real*, but rather *expresses a model of reality in a form convenient for calculation.*
- A consequence of "mathematics as model" is that you must always carefully consider whether you (or the computer!) are doing the right calculations.
- In other words, your contribution to a project is not the *numbers* you present, but rather the *choice* of calculations to do, and the assurance that those calculations are *appropriate.*
  - If you're the programmer or data engineer, you're also responsible for ensuring the calculations are *correct.*

What is mathematics? An applied view

In *Software for Data Analysis: Programming with R*, J. M. Chambers describes statistical software. But the same point applies to using mathematics as it does to programming.

> [Many readers] will have some experience ... with software for statistics, but will view their involvement as doing only what's absolutely necessary to "get the answers". This book will encourage moving on to think of the interaction with the software as an important and valuable part of your activity. You may feel inhibited by not having done much programming before. Don't be. Programming ... can be approached gradually, moving from easy and informal to more ambitious projects. As you use [software], one of its strengths is its flexibility. (pp. vii–viii)

Substitute "mathematics" for "programming" and "software".

What is mathematics? A mathematician's view

- Prof. Saunders Mac Lane wrote a 456-page book to answer that question, in which he concluded:

  > Mathematics aims to understand, to manipulate, to develop, and to apply those aspects of the universe which are formal.

- By "formal aspects of the universe" Mac Lane means something like "those things which can be calculated." But *calculation* itself is only a small part of Mathematics to Mac Lane.

- *Understanding* when a calculation is applicable, *manipulating* algorithms to produce new ways of calculating, and *developing* more applications where calculation is useful is the major part.

What is mathematics? I'll say that again

- Math is the formal treatment of variables and their relations.
- This can be complex, so people often think that "math is hard."
- That's incorrect: the *problems* we model mathematically are hard, and mathematics makes solving them (*correctly!*) *easier*.
- The most basic and oldest math treats variable values, specifically numbers, and the arithmetic operations.
- These generally correspond to things we can measure by eye (length) or count (coins), and so can be treated with simple math, like "arithmetic".

Formal analysis

What is “formal analysis” (or “formal treatment” as mentioned above)?
Saunders Mac Lane says that the essence of the use of mathematics is to

- express a real-world problem as an equation or expression,
- manipulate the syntax of the equation according to certain rules, and
- then interpret the final form as the solution in real-world terms.

What mathematicians do

- We have some *real* mathematicians on the Shako faculty, for example Professors Hachimori and Yoshise.
- But most of us aren't mathematicians, and even the mathematicians' students frequently don't do "real" mathematics.
- What do I mean by "real" mathematics, and why should you care?

What mathematicians do: Foundations of arithmetic

Mathematicians generally use axiomatic systems to perform formal analysis. Simple mathematics, such as arithmetic, uses simple systems. Here are the first three Peano Postulates describing what we call the *natural numbers*, $\mathcal{N} = \{0, 1, 2, \ldots\}$:

1. 0 is a number.
2. Every number n has exactly one successor, n'.
3. Two different numbers $m \neq n$ have different successors, $m' \neq n'$, and *vice versa*.

Obvious enough?

More about Peano's postulate system: `https://en.wikipedia.org/wiki/Peano_axioms` (Japanese and Chinese versions available).

Finite number systems

- But these aren't enough to characterize everyday arithmetic.
- Let 1 be the successor to 0 and 0 the successor to 1. There is no contradiction to the postulates, and arithmetic can be performed according to the tables:

$+$	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

Figure 1: Boolean arithmetic

Finite and trivial number systems

- It's even "better" than the natural numbers since subtraction "just works":

$-$	0	1
0	0	1
1	1	0

- Division by zero still fails and there's no useful way to make it work. We can make it work by letting $0' = 0$:

$+, -, \times, \div$	0
0	0

So what?

- The 1-element number system $\{0\}$ is useless in daily life.
- The 2-element number system $\{0, 1\}$ is not familiar to us, but it underlies formal logic and all modern computer hardware operations. It's called a "Boolean algebra."
- We *do* use n-element number systems $\{0, 1, \ldots, n-1\}$ every day!

The natural numbers

- The natural numbers also fit this scheme: 0 *is* a number, every number n *has* a successor $n + 1$, and two numbers $m \neq n$ differ *if and only if* their successors $m' \neq n'$ differ.
- What makes the natural numbers special? There is a fourth axiom:

  4. 0 is not the successor of any (natural) number.

  This prevents any cycle, and the existence of a successor to every number means that $\mathcal{N}$ is not finite.

What about arithmetic?

- Where do the numbers 1, 2, 3, ..., come from? Postulate 1 says 0 is a number, but no postulate says anything about 1! The answer is simple. Define:

$$1 = 0',$$

and that's all. 1 is just another name for $0'$, 2 is just another name $1' = 0''$, and so on.

- **Much of the work of mathematicians and others who use mathematics is defining convenient abbreviations for tedious notation and operations.**

- For example, we don't need to say anything more about number to put addition into our number system. We just *define* addition as follows:

$$\begin{aligned} a + 0 &= a \\ a + b' &= (a + b)' \end{aligned}$$

- Thus, $a + 1 = a + 0' = (a + 0)' = a'$.

What about the integers?

- OK, we've defined addition, but we have the problem that the inverse of addition is not well-defined. The formula 2 - 4 doesn't have a value in $\mathcal{N}$. Do we need to add, say, -1 to our numbers with a new postulate? Or perhaps the idea of "predecessor"?
- No! We can *define* the integers as the *differences* of natural numbers.
- Is 2 an integer? Yes, because we can write it as 4 - 2, which is a difference of natural numbers.
- What is 2 - 4, then? It's an integer. It's the same integer as 0 - 2. (This can be proved, but it's tedious and may require a more careful definition, so I'm not going to try.) We can choose 0 - 2 as the standard way to express this integer, and in fact we have: 0 - 2 = -2, of course! -2 is just a more convenient way to say 0 - 2.
- We now have a system of numbers with additive inverses (*e.g.*, 2 + (-2) = 0).

What does it all mean?

- Mathematics is not the hard way to do things unless you don't care about getting things right. (Yes, there are problems you can't use math to solve. Those are the hardest problems to get right!)
- Mathematics allows generality. The three postulates for the usual natural numbers also can be used to prove things about finite number systems.
- Mathematics can help to understand how problems differ. The natural numbers are special because 0 is not the successor of any (natural) number.
- Much of mathematics just discovers simple rules that are shortcuts for tedious computation (the usual addition tables *versus* counting successors).
- You don't prove the rules yourself. Most uses of mathematics involve defining expressions to model your real-world problem, then solving the formal model mathematically, then reinterpreting the results as real-world actions.
- Your real-world problem has to match the mathematical rules or your computations will give *unreal* answers.

Homework 1, due May 8, 12:00 noon

Read and understand the following instructions on submission of homework. **If you do not follow them, you will not receive credit.**

Submit this assignment by *email*. Give the mail the subject ”01CN101 Homework #<number> by <your name>” in *hankaku romaji* and send it to `turnbull@sk.tsukuba.ac.jp`. (This subject is necessary for automatically sorting incoming mail.) It should look like this:

```
Subject: 01CN101 Homework #1 by Stephen Turnbull
```

for Homework #1.

Make sure that the body of the email contains your *name* and *student ID number*.

If you are late, submit the assignment for partial credit. The later, the less credit you will receive. If you believe that the late submission is in part due to lack of care by the instructor, or some event (such as hospitalization) required your full attention for two full days or more, you may explain for additional credit. Otherwise, I don't care why it was late.

Homework 1(i) Review

Here are the first three Peano Postulates describing what we call the *natural numbers*, $\mathcal{N} = \{0, 1, 2, \ldots\}$:

1. 0 is a number.
2. Every number n has exactly one successor, n'.
3. Two different numbers $m \neq n$ have different successors, $m' \neq n'$, and *vice versa*.

The natural numbers constitute an *infinite set*, but there are also finite number systems that satisfy these axioms, such as the Boolean algebra on $\{0, 1\}$. These number systems are characterized by the number of elements, n. (The Boolean algebra has $n = 2$.)

Homework 1(ii) Problems

1. How do we use the following finite number systems in daily life? You may want to rename numbers to something more familiar. Some systems have more than one use!

$$n = 7, \qquad n = 12, \qquad n = 24, \qquad n = 60, \qquad n = 360.$$

2. Present any other examples of finite natural number systems you can think of (you may stop at 5 if you can think of so many).

3. The analog of finite natural number systems is *bounded* sets of *real* numbers. Present as many examples of *bounded* real number systems that we use as you can (you may stop at 5 if you can think of so many).

Errata

I've made a number of corrections, mostly very minor, in the "as delivered" version of Lecture 1 Notes. The following are worth drawing attention to.

p. 24 In the phrase "not a good model" in the last item, it's important to remember that models are *purposeful*, we have reasons for building them the way we do, and different purposes require different models, because they require different abstractions (what you can and cannot leave out). I add a note about the purpose for the evaluation "not a good model."

p. 25 The original was in bad style, mixing equations *without* the time argument ("(t)") and those *with* it. Now all equations have it.

Appendix A: Proof that $2+2=4$

It's not just a table somebody made up.

1. First, we translate from our convenient names to the "names" that the Peano Postulates use. "2" $\to$ "$0''$" and "4" $\to$ "$0''''$", giving

   To prove: $0'' + 0'' = 0''''$.

   This is the crucial "trick" of many mathematical proofs: restate the problem in a form that fits the syntax rules.

2. Now it's just a calculation:

$$\begin{aligned} 0'' + 0'' &= (0'' + 0')' \quad \text{Rule 2} \\ &= (0'' + 0)'' \quad \text{Rule 2} \\ &= (0'')'' \quad \text{Rule 1} \\ &= 0'''' \end{aligned}$$

Parentheses are special

- In the last step we "just dropped" the parentheses. Can we do that?
- We can, because they aren't really part of the formula. In the earlier lines they were needed to show which "successor" operations were parts of the numbers being added, and which were being applied to the result of the addition.
- This may be clearer if we use functional notation: $s(n) = n'$ and $a(n, m) = n + m$.
- Then the rules are

$$\begin{aligned} a(n, 0) &= a \\ a(n, s(m)) = s(a(n, m)) \end{aligned}$$

- and the proof is

$$\begin{aligned} a(s(s(0)), s(s(0))) &= s(a(s(s(0)), s(0))) \quad \text{Rule 2} \\ &= s(s(a(s(s(0)), 0))) \quad \text{Rule 2} \\ &= s(s(s(s(0)))) \quad \text{Rule 1} \end{aligned}$$

- The “trick” is that, in functional notation, in Rule 1, one pair of parentheses is removed, while in Rule 2, the number of parentheses stays the same.

The heart of math is syntax manipulation

- I just realized that we can "fix" the "usual" algebraic notation by *requiring parentheses for addition* in the definition's rules:

$$\begin{aligned} (a+0) &= a \\ (a+b') &= (a+b)' \end{aligned}$$

  See how Rule 1 automatically removes parentheses, but Rule 2 preserves them? (I wonder: do most students even see the differences between these rules and the ones presented in lecture?)

- The central idea is that you don't need to know what the operations and equations mean to do calculations with the definition of addition. You just need to be able to recognize patterns like "(" **thing** "+ 0)", and using Rule 1, *copy* all the characters up to that "(", leave out the "(", copy **thing**, leave out the "+ 0)", and finally copy the rest of the characters! There's no "math" here!

- Or, following Prof. Mac Lane, this is all the math there is.

Appendix B: More about logistic growth

- In lecture I showed the similarity of logistic growth to exponential growth using a graph.
- We can also do this with algebra:

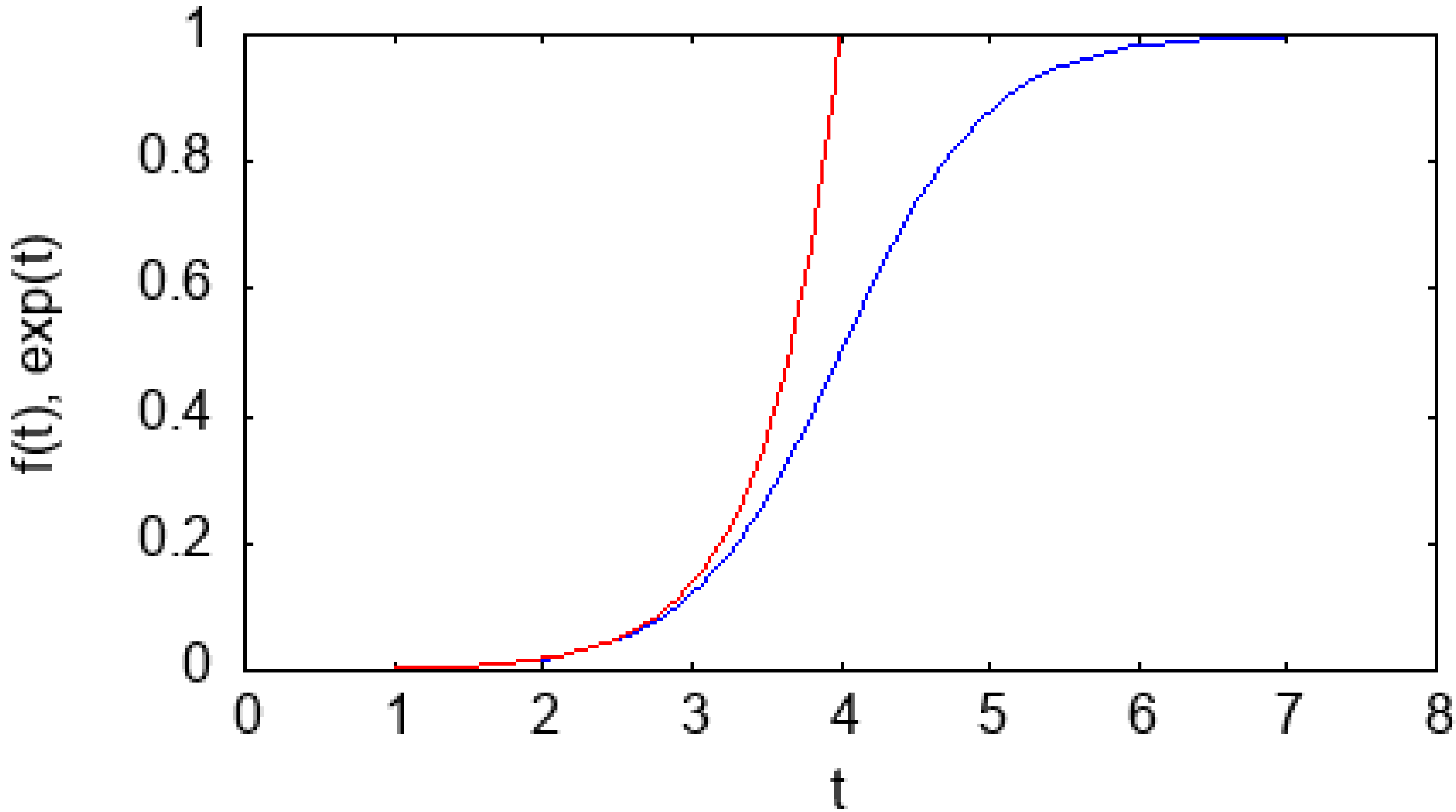
$$\begin{aligned} f(t) &= \frac{e^t}{e^t + e^{-t}} \\ &= \frac{e^{2t}}{e^{2t} + 1} \end{aligned}$$

- It now may be "obvious" to you, but it's OK if it's not. Because $e^{-4} \approx 1.83\%$ is very small, the denominator is very close to 1. So at $t = -4$, $f(t)$ is very close to e^{2t}. Furthermore, because e^{-4} is so small, the denominator is changing very slowly, so $f(t)$ *stays* very close to e^{2t}. (There is math for these English statements, of course.)
- Amusingly enough, this exercise allowed *me* to generate a much better graph! See! I told you! Math makes things *easier* and *better*. (And *faster*, although that probably has a lot to do with the fact that I like math and do

a lot of it.)

Logistic *vs.* exponential, again

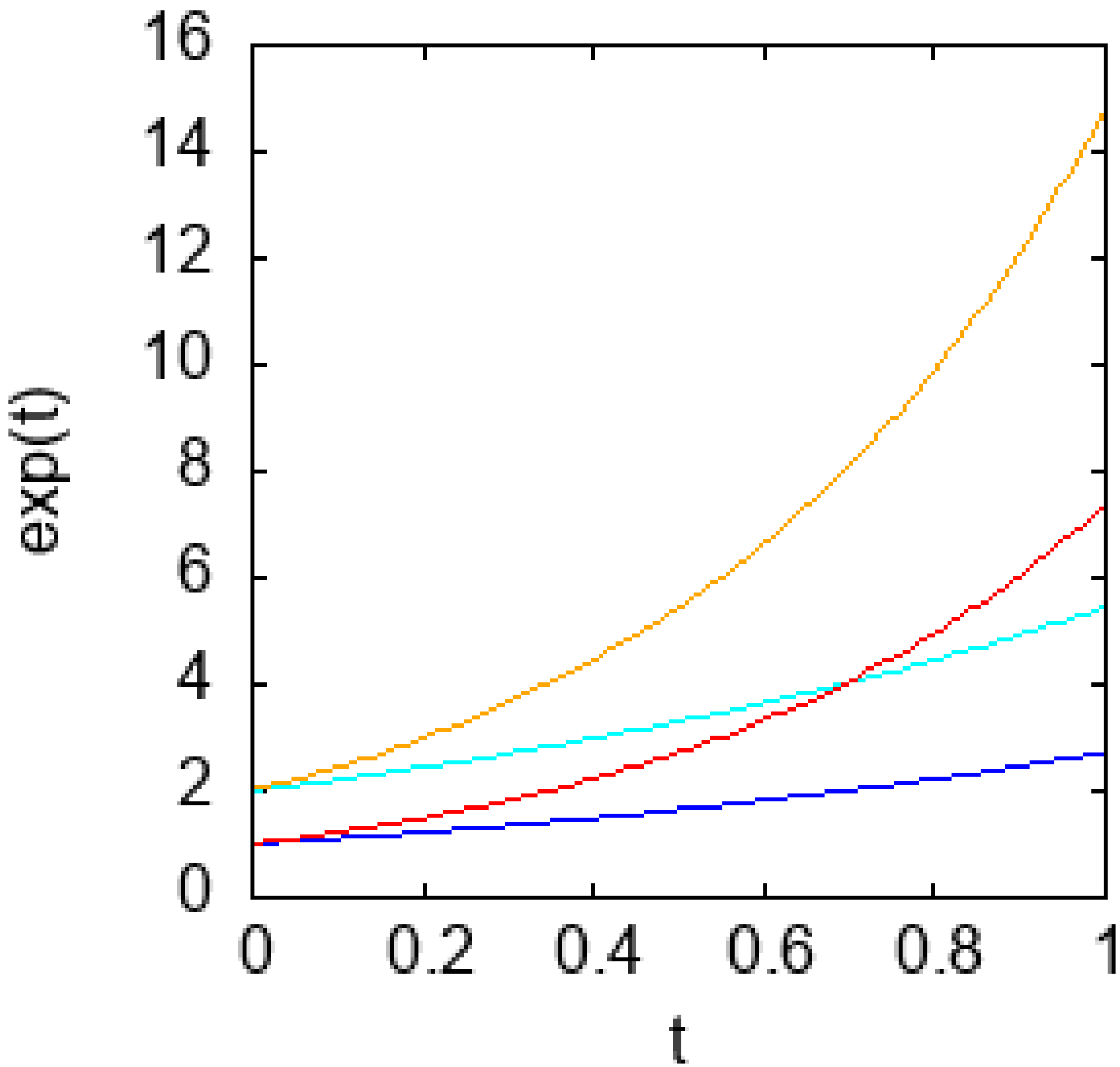
$$f(t) = \frac{e^{t-4}}{e^{t-4}+e^{4-t}} \qquad \exp(t) = \frac{e^{2t-4}}{e^4}$$



Maybe you're still not convinced, but think about doing more math, OK?

Appendix C: More about logistic growth

In the lecture notes, the title promised "graph(s)," but I gave you only one. Here are the missing graphs. First, the generic exponential function is $Ae^{\alpha t}$, where *the* exponential function e^t has $A = \alpha = 1$. These are the four exponentials with $A = 1$ or $A = 2$, and $\alpha = 1$ or $\alpha = 2$.



16
14
12
10
8
6
4
2
0
exp(t)
0
0.2
0.4
0.6
0.8
1
t

Now, on split-screen