# Economic Dynamics

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## Abstract

Stochastic processes.

# Stochastic processes

- A *random variable* is just a function $x : Omega \to X$ from a *probability space* $(\Omega, \mathcal{F}, \mu)$ to some other space $X$.
- Think of the probability space just as a "source of randomness." Mathematically it's somewhat complex, but we don't need the complexity here.
  - The reason for the complexity goes back to the example showing that even with a huge amount of flexibility in defining outcome sequences, there were only two possible outcomes of the iterated function. We need "big infinities" here to give "enough" flexibility to our stochastic processes.
- The other space can be a space of *sequences* or *functions* over time, called the *realization space.*
  - We can also define stochastic processes in *space.* For example, the geographical distribution of earthquakes in Japan.

# Types of stochastic processes

**discrete** In a *discrete time process* (from now on we frequently omit "stochastic"), the time variable is the integers (typically the nonnegative integers, sometimes all integers, and sometimes a finite segment of the integers).

**continuous** In a *continuous time process*, the time variable is an interval of the real line, possibly infinite.

**continuous** In a continuous time process, the realizations may be continuous. This is the implied meaning of "continuous" (continuous time continuous process).

**jump** In a continuous time process, the realizations may be continuous *almost everywhere*, except at a countable set of times where the process jumps discontinuously.

- A countable set can be *dense* in the time interval, so that's a lot of jumps.
- It's hard to even imagine what a process that isn't continuous almost everywhere would be like.

# Examples

**Brownian motion** The most famous stochastic process, *Brownian motion* is a continuous time continuous process. It is the basis for most of modern financial market theory. It has the spectacular property that although it's continuous, it is nondifferentiable almost everywhere.

**random walk** The basic random walk is a discrete time process with realizations that are sequences of integers, with $x_0 = 0$, and the difference between outcome and its successor is either +1 or -1. A useful example, which can be used to simulate Brownian motion.

**Poisson counting process** A jump process whose values are integers. Extremely useful in *queuing theory*.

# Markov processes

- As with (nonstochastic) dynamic optimization theory, we can also consider stochastic processes as recursive processes or (stochastic) differential processes.
  - For example, you probably found the description of the *random walk* a bit fuzzy (and if you already know what a random walk is, you'll know it's incomplete).

    It's probably easier to understand if it's defined as a sequence of states $x_t$ and a sequence of independently identically distributed variables $\epsilon_t$ which take the values +1 and -1 with equal probability, and $x_{t+1} = x_t + \epsilon_t$.
- *Brownian motion with drift* is defined as a continuous time process with independent increments $\Delta(t-s) \equiv x(t) - x(s)$ where $\Delta(\tau)$ is distributed normally with mean $\tau\mu$ and variance $\tau\sigma^2$.