Economic Dynamics

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Abstract

Market stability.

Complex dynamics. See complex_dynamics.pdf and complex_dynamics.ipynb (requires Python 3.6, NumPy, Bokeh, Jupyter).

Discrete dynamic systems

- A *discrete dynamic system* is a sequence of values (scalar or vector) $\{y_t\}$ defined recursively. (The braces indicate that it is a sequence of y_τ for $\tau = 0, \ldots, t$.)
- *Recursive* means there is a *functional relationship* $y_{t+1} = f(\{y_t\})$ for $t = 0, 1, 2, \ldots$.
- An alternative expression for a discrete dynamic system is a *difference equation* $\Delta y_{t+1} \equiv y_{t+1} - y_t = g(\{y_t\})$.
- Clearly, we can always define a recursive equation from a given difference equation by $f(\{y_t\}) = g(\{y_t\}) + y_t$, and vice versa.
- Difference equations are closely related to differential equations, and are used in computational simulation of differential equations.

Classifying Difference Equations

- As with differential equations, we can take differences of differences:

$$\Delta^2 y_{t+2} = \Delta y_{t+2} - \Delta y_{t+1} = (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t)$$

  The exponent on Δ is called the *order* of the difference equation.

- As with differential equations, a higher-order difference equation can be broken down into a system of first-order difference equations.

- If f is linear, the difference equation is called *linear*.

- In general, f can depend on t. If it does, the dynamic system is not *autonomous*.

- We consider only the autonomous case where f is the same function of y for all t. This is more than a matter of simplifying computation and notation: it rarely makes sense to discuss *steady states* in non-autonomous systems.

- We classify recursive function systems in the same way.

Difference Equations and Polynomials

- Consider a dynamic system where the position of a falling object is measured with respect to time. Suppose it started at rest at a height of c.
- Then its position at time t is $x_t = c - at^2$, where $a = \frac{1}{2}g \approx 4.9m/s^2$.
- Suppose the position is measured at one second intervals.
- Then the first difference is $\Delta x_t = x_t - x_{t-1} = (c - at^2) - (c - a(t-1)^2) = a - 2at$.
- The second difference is $\Delta^2 x_t = \Delta x_t - \Delta x_{t-1} = (a - 2at) - (a - 2a(t-1)) = -2a$.
- Thus a second-order difference equation based on a quadratic time path is always a constant.
- Note that this analysis is based on the length of the interval analyzed being a constant.

A Recursive Equation Example

- Consider an infinite sequence of binary choices. We represent one alternative by "0" and the other by "1".
- An example sequence could be written "$z = .1101000110101\ldots$", suggesting the binary representation of real numbers between 0 and 1. In fact, this is the case, and there is a continuum of such sequences (a "big" infinity). Note: $z_0 = 1$, $z_1 = 1$, $z_2 = 0$, ...
- Consider the (non-autonomous!) dynamic system defined by a sequence of functions f_t such that $x_t = f_t(x_0, \ldots, x_{t-1})$ for $t = 1, 2, \ldots$. Evidently we can derive a new infinite sequence from any infinite sequence by applying this system to it. Call this sequence $f(z)$ (no subscripts!)
- Question: how big is the set $\{f(z) | z \in [0, 1)\}$?

The Surprising (?) Answer

- You might think that with all the flexibility of an infinite sequence of functions depending on everything in the past and a big infinity of z to choose from, the answer would be something like "a lot."
- In fact, the answer is **two**.
- The only part of z that affects the result sequence x is the *first* component z_0:

$$\begin{aligned} x_0 &= z_0 \\ x_1 &= f_1(x_0) \\ x_2 &= f_2(x_0, x_1) \\ x_3 &= f_3(x_0, x_1, x_2) \\ &\vdots \end{aligned}$$

The Surprising (?) Answer, *cont.*

- This sequence can be rewritten

$$\begin{aligned}
x_0 &= z_0 \\
x_1 &= f_1(z_0) \\
x_2 &= f_2(z_0, f_1(z_0)) \\
x_3 &= f_3(z_0, f_1(z_0), f_2(z_0, f_1(z_0)) \\
&\vdots
\end{aligned}$$

- $x = \{x_0, x_1, x_2, \ldots\}$ depends only on z_0, which must be either 0 or 1!
- In fact, the answer is **two**.
- From now on, we consider only sequences of scalars, and the case where the $\{y_{t-1}\}$ are ignored. The general case isn't harder in principle, but the computation and notation are tedious.

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Discrete Processes

Some interesting processes really are discrete. Examples:

- The cobweb, Hicks, and Marshall adjustment processes can be considered as the result of (rather naive) behavior in a series of markets. For example, agricultural markets from one year to the next. (It is the inelasticity of demand and supply for food that made Marshall decide to use a model where quantity *adjusts* and price *responds to the quantity adjustment*.)
- Human beings cannot be continuously active. They need to sleep, and function best on a consistent daily sleep cycle. Thus it makes sense to measure human activity per day.
- Most activities consist of a sequence of tasks. The tasks may take variable amount of clock time to complete, but it often makes sense to count "time" in terms of tasks completed rather than watching the clock.
- Newton's method for solving equations.

Discrete Simulations

But one of the most important uses of discrete processes is *simulation* of continuous processes.

- You've probably used the *Artisoc* or *Netlogo* software in jisshuu to model crowd behavior in buildings or in disaster response.
- Weather simulations: global warming, general local weather, and specific phenomena like typhoons or El Niño.
- Numerical computation of processes such as price adjustment in a market.

Example Simulations

- The Walras price adjustment process.
- The Solow growth model.
- The fishery.

Stability

- The general idea of stability in dynamics is
  - A dynamic process is "stable" if it tends to an equilibrium.
  - An equilibrium is "stable under a process" if the process tends to the equilibrium.
- The definitions for markets generally suppress the process, and distinguish between local and global stability.
  - A solution of a model is *locally stable under a process* if the process converges to the equilibrium started at any point close enough to the equilibrium.
  - A model is *globally stable under a process* if the process converges to some solution no matter where it is started.
- In case of a market equilibrium, the definitions of *solution* (equilibrium) and *process* are independent of each other. In the case of a growth model, *solution* is defined as a steady state of the *process*.

Finding the Equilibrium

- Most microeconomic analysis simply assumes markets are in equilibrium.
- But in principle we need to worry about two issues:
  - After a major change in the environment (*e.g.*, a new market), how can the market find the equilibrium?
  - If the price is a little bit uncertain, will it tend to stay in the region of equilibrium, or wander away?
- The first is determined by the condition called *global stability*, the second by *local stability*.

Why Call It Stability?

- The two tasks (finding equilibrium and staying in equilibrium) might seem to be different in principle.
- However, both are dynamic processes, and in the analysis of both, environmental conditions are assumed to be unchanging once the process starts.
- Thus both dynamic processes are dependent on market price and quantity, and perhaps on the history of the process.
- *Formally*, the processes are indistinguishable, except that "non-wandering" is assumed to take place "near" the equilibrium (thus *local* stability, in the usual ϵ-δ sense of calculus), whereas "discovery" may have no historical price or quantity information to work with (thus *global* stability).
  - Global stability requires convergence to *some* equilibrium (there may be several) for any initial conditions of the process, while local stability may require initial conditions to be "sufficiently near" a particular equilibrium.

Defining Stability

- There are multiple definitions of stability, each of which is characterized by an *adjustment process* which dynamically adjusts price and quantity, based on current conditions.
- An equilibrium is *stable* if the adjustment process converges to the equilibrium.
  - It is *globally stable* if convergence occurs for *all* initial conditions in the domain of the adjustment process.
  - It is *locally stable* if convergence occurs for all points in a region of the domain around the equilibrium, where the equilibrium is not on the boundary of the region.
- There are many possible adjustment processes. Both local and global stability are defined only for a particular adjustment process, which must be mentioned when saying that an equilibrium is stable.
- An equilibrium that is globally stable necessarily is locally stable (with the region being the whole domain!)

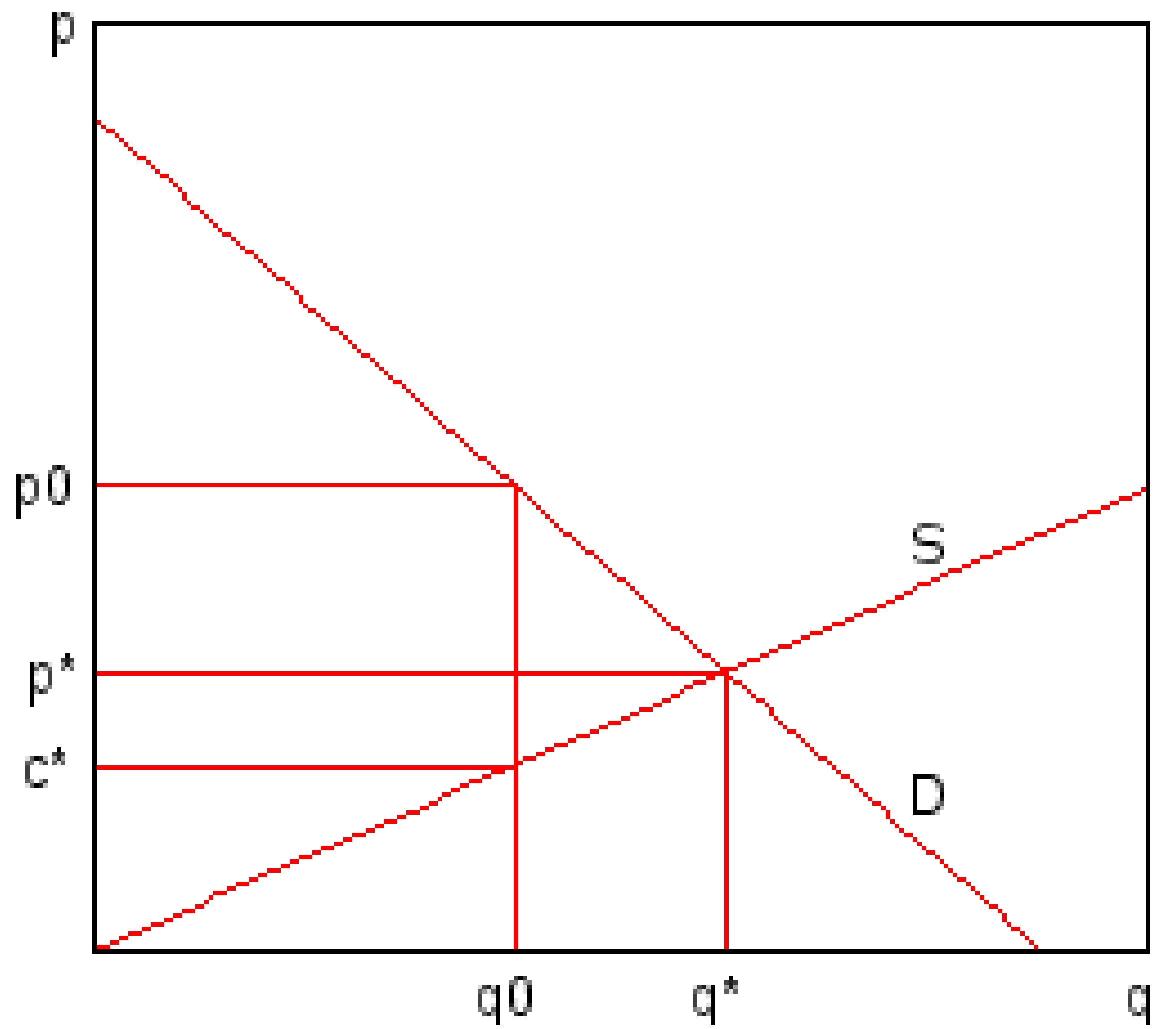
Defining an Adjustment Process

- An adjustment process is a function mapping information about the state of the process either to a new state (for discrete time) or to its derivative (continuous time).
- We need to pick the domain of the function ...
- ...and the function itself.

A Simple Adjustment Process

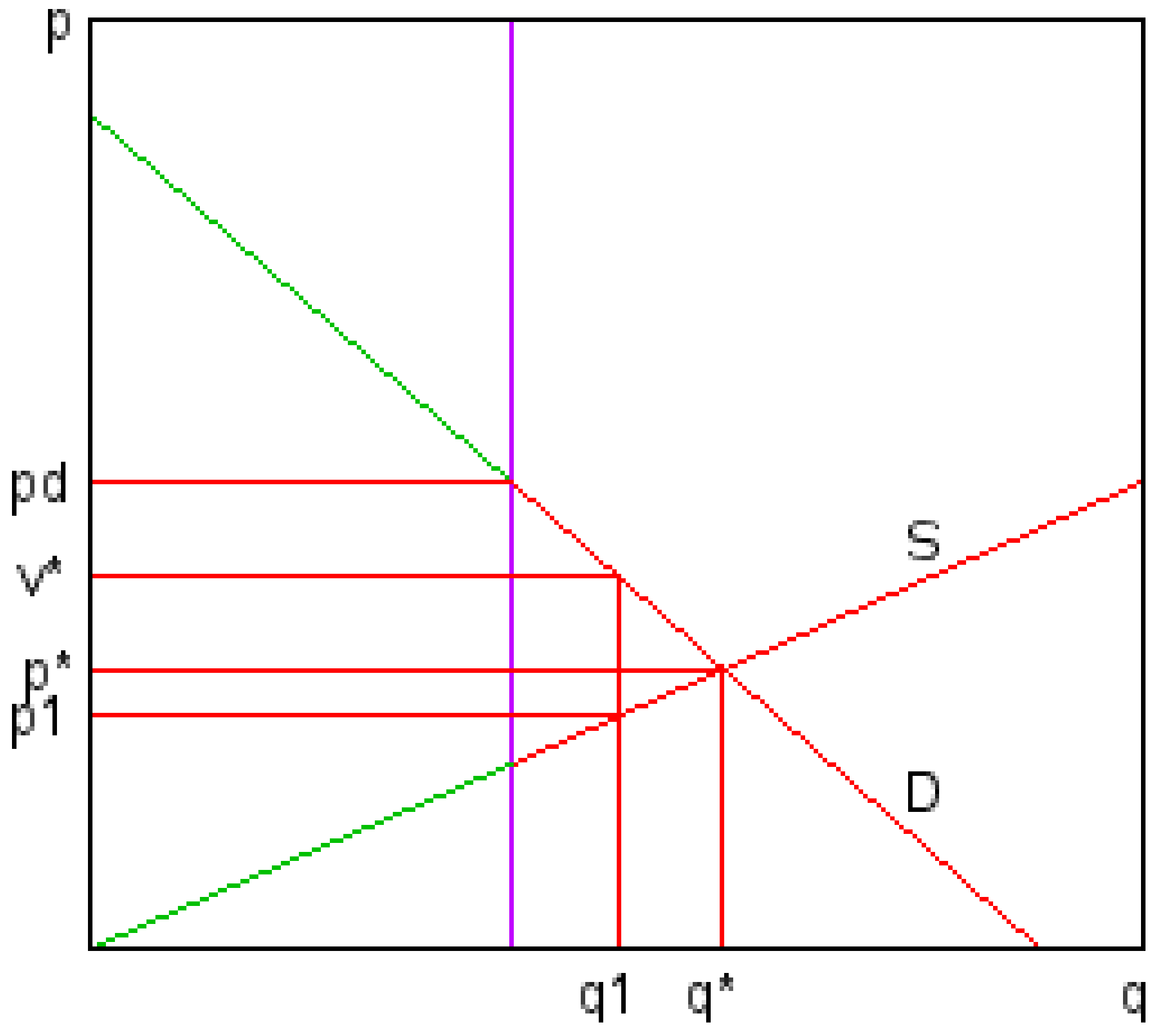
- Assume we have a large number of buyers and sellers, each wishing to trade one unit of a good if the price is advantageous. Denote the resulting demand and supply curves by $D(p)$ and $S(p)$, and define *excess demand* as $Z(p) = D(p) - S(p)$.
- The domain is the price.
- Our adjustment process proceeds by picking prices as follows:
  1. Set $p_s = 0$ and $p_d = \infty$.
  2. The initial price p_0 is picked at random.
  3. $q_t = \min\{D(p_t), S(p_t)\}$; all buyers with value $v \geq D^{-1}(q_t)$ and all sellers with cost $c \leq S^{-1}(q_t)$ trade, and leave the market.
  4. If $Z(p_t) = 0$, stop; p_t is the equilibrium price.
  5. If $Z(p_t) < 0$, set $p_d = p_t$, otherwise set $p_s = p_t$.
  6. If $p_d = \infty$, set $p_{t+1} = 2p_t$, otherwise set $p_{t+1} = \frac{p_d + p_s}{2}$, and go to step 3. ($p_d = \infty$ is a special case, and can only occur if $D = 0$.

Simple Process: Initial Condition



Linear demand and supply curves with $p^* < p_0 < a$.

Simple Process: First Adjustment



Linear demand and supply curves with $p^* < p_0 < a$.

Is This a “Good” Process?

- This process always converges to equilibrium. That’s good!
- But the assumption that trades take place at each price, and the extreme value agents do all the trading is hard to justify.
- Especially so in a case of a good where agents trade multiple units. Why? Because the curves change due to income effects, so the computations are incorrect. (The correct computations would still converge to equilibrium, but are much more complicated.)
- Also, in either case, the traders on the long side of the market would benefit from waiting until the next adjustment (the terms of trade always improve for them).
- Both problems can be avoided by use of a tâtonnement process (using “fictitious trading” as in the cobweb process).

Tâtonnement

- The term *tâtonnement* (“groping”) was introduced by the French economist Leon Walras (a contemporary of Karl Marx), generally considered the earliest general equilibrium theorist.
- Although there is some question about what Walras intended, today a *tâtonnement process* is defined as one in which no trade is made at disequilibrium prices.
- Both problems mentioned previously are obviously avoided, since they depend on actual trades being made out of equilibrium.
- Tâtonnement is generally considered to be unrealistic, but (outside of controlled experiments), not only do agents trade out of equilibrium, endogenously changing supply and demand, but exogenous parameters that affect supply and demand are also changing.
- If our stability analysis abstracts from the “moving target” nature of equilibrium, tâtonnement doesn’t seem too much harder to swallow.

Adjustment processes for a market

- In the *cobweb model*, time is discrete, and supply responds to the past price while demand responds to the current price.
- In Hicks and Marshall stability, the adjustment process is left abstract, and the stability properties analyzed in terms of “arrow diagrams”.
  - This works for a single market because they imply a single differential equation.
- Walrasian stability is like Hicks stability, except that the differential equation is explicit, and worked out for more general cases.

The Cobweb Model

- Partial equilibrium (a single market).
- Demand responds to the current price in the market.

$$q_t^d = D(p_t)$$

- Supply in the current period is inelastic because of a production lag. It is set according to the previous period's price. (Period is defined by the production lag.)

$$q_t^s = S(p_{t-1})$$

- Equilibrium, as usual, clears the market:

$$q_t^s = q_t^d$$

- This generates an iterated function system:

$$p_t = D^{-1}(S(p_{t-1}))$$

Cobweb Example

- Take a simple linear supply and demand system:

$$\begin{aligned} D(p) &= a - bp \\ S(p) &= c + dp \end{aligned}$$

- Substituting in the iterated function system gives

$$\begin{aligned} p_t &= \frac{a-c}{b} - \frac{d}{b} p_{t-1} \\ &= \frac{a-c}{b} \sum_{s=0}^{t-1} \left(-\frac{d}{b}\right)^s + \left(-\frac{d}{b}\right)^t p_0 \end{aligned}$$

Now $\lim_{t\to\infty} \sum_{s=0}^{t-1} \left(-\frac{d}{b}\right)^s = \frac{b}{b+d}$, so the first term in the expansion of p_t tends to $\frac{a-c}{b+d}$ as $t \to \infty$, and the second term converges to zero if and only if $\left|-\frac{d}{b}\right| < 1$, *i.e.*, $|d| < |b|$. That is, supply is less elastic than demand.

Parametric stability conditions

Need a table here.

The Cobweb Dynamic

1. This adjustment process starts by assuming that firms set a common price.
2. The consumers respond by announcing demand quantity.
3. Then the firms respond by setting price again, to elicit the quantity they want to supply at the current price.
4. Repeat steps 2 and 3 until convergence.
5. Because we start and end with price in each cycle, the domain of the process is the price axis. Recall that the *inverse supply function* is $S^{-1}(P) = \{Q \mid Q = S(P)\}$, where $Q = S(P)$ is the supply function. Then we can write the equation

$$P_{t+1} = S^{-1}(D(P_t)),$$

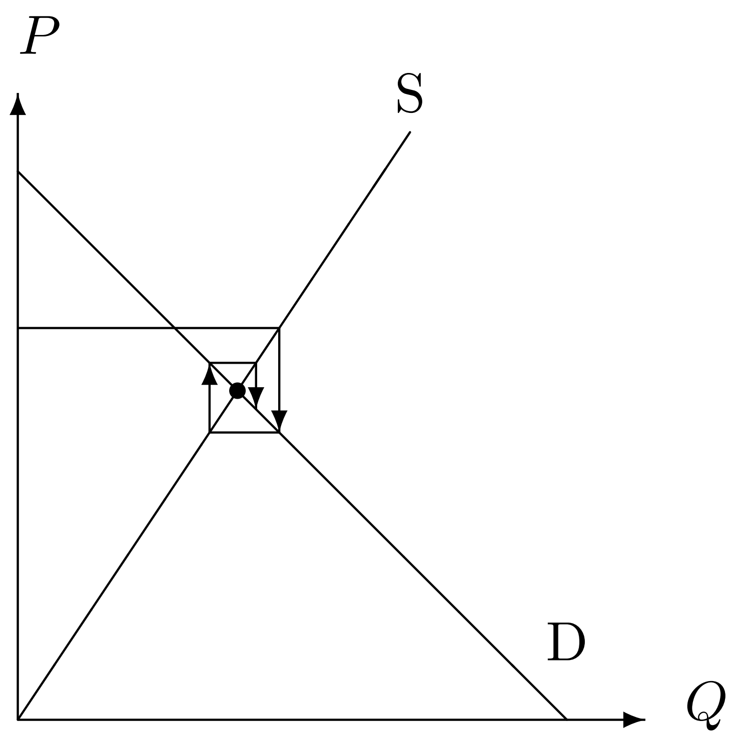
for the *cobweb adjustment process*, where $Q = D(P)$ is the demand function.

The Cobweb Dynamic

At each price, the supply side determines the quantity. The excess demand causes fierce competition among the members of the long side of the market, inducing a new price.

"Fictitious trading:" no trades actually occur until equilibrium price is found.

Requires a slope condition or there is no convergence.

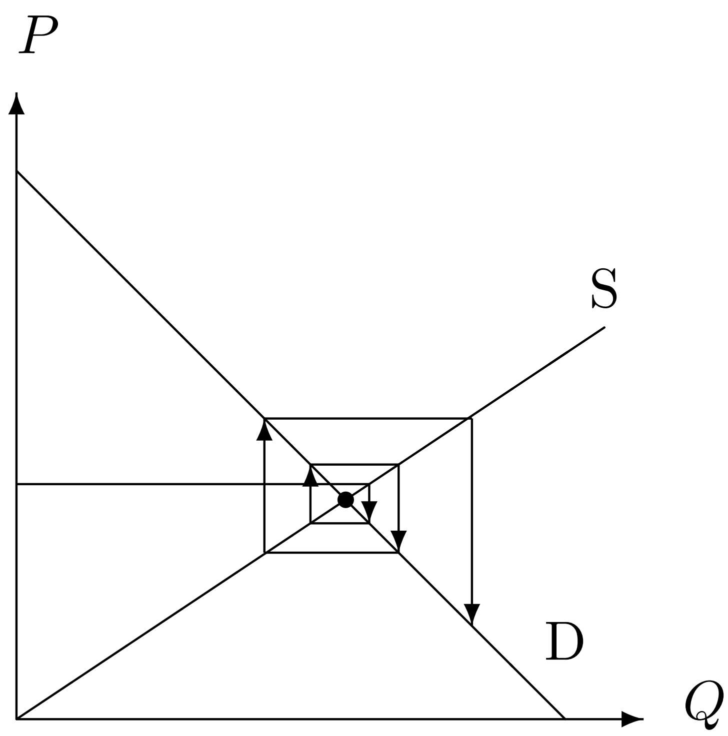


Problems of the Cobweb Dynamic

- Firm behavior is very unrealistic. To compute the next price, the firm needs to know not only the market supply function (perhaps reasonable, since other firms are likely to have similar marginal cost), but also the market demand function. If it knows all that, why doesn't it go directly to the equilibrium?

- The cobweb adjustment process fails to converge when $0 < -D'(P) \leq S'(P)$. (Notice that this means that supply is upward-sloping and demand is downward-sloping as usual, and that demand is *steeper* than supply on the usual graph.)

Nonconvergence in the Cobweb Dynamic



- Occurs when supply is more elastic (flatter on the conventional graph) than demand.

Marshall and Hicks Stability

- Marshall and Hicks did not fully define adjustment processes.
- Similar to the cobweb process in that current price and quantity are used to compute the next approximation on one of the market-defining curves.
- They differ in whether they are quantity adjustment processes or price adjustment processes.
  - In Marshallian stability, if previous period's quantity induced excess demand (the demand price is higher than the supply price) the quantity is increased, while quantity is reduced when there is excess supply.
  - In Hicksian stability, if there was excess demand at the previous price (quantity demand greater than quantity supplied), the price is increased, while it is decreased for excess supply.
- It is assumed that the step is small enough that eventually the process converges even if it overshoots sometimes.
- Both processes are stable with upward-sloping supply and downward-sloping demand, but can be unstable with "perverse" demand and/or supply.

Walras' Adjustment Process

- The Walras adjustment process is a continuous time process that is more realistic than the discrete-time processes described above:
  - It approximates small adjustments leading to equilibrium.
  - The cobweb process requires the firm to know the whole demand curve, Walras only the "local" condition of the market at the current price.
  - It "aims at" equilibrium, while the "short side adjusts" processes assume current conditions will persist despite planned changes.
- It's very simple to express as a mathematical formula. Define the *excess demand at price p* as $z(p) = D(p) - S(p)$. (Note that excess supply is represented as negative excess demand.) Then the continuous-time Walras adjustment process is just

  $$\dot{p} = kz(p).$$

  $k > 0$ is the "speed of adjustment" parameter, usually taken as $k = 1$.

Properties of Walras' Adjustment Process

- For $z'(p) < 0$ (globally *downward-sloping excess demand*), we have a *negative feedback loop* and the market is globally stable with a unique equilibrium under Walras' process.
  - If demand is downward-sloping and supply upward-sloping in the usual way, excess demand will be downward-sloping.
- In a multimarket model (including general equilibrium), Walras' process again gives stability if all markets have downward-sloping excess demand and satisfy a condition called *weak gross substitutes* on cross-price elasticities.