Economic Dynamics

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Abstract

The economics of pure exhaustible resources, basic concepts of dynamic optimization, and examples from the fishery.

Pure Exhaustible Resources

- Abbreviate *pure exhaustible resource* to *exhaustible resource.*
- Exhaustible resources are *rival* goods.
- They may be non-excludable (ocean fishing), partially excludable (large oil fields), or completely excludable (small oil fields).
  - Socially optimal usage patterns don't vary; they depend only on the stock.
  - Degree of excludability helps determine market structure, and the equilibrium usage patterns are different.
- The basic ideas of dynamic optimization can be seen with minimum technical difficulty in a model of a monopoly business which owns the whole stock of a resource.

Monopoly

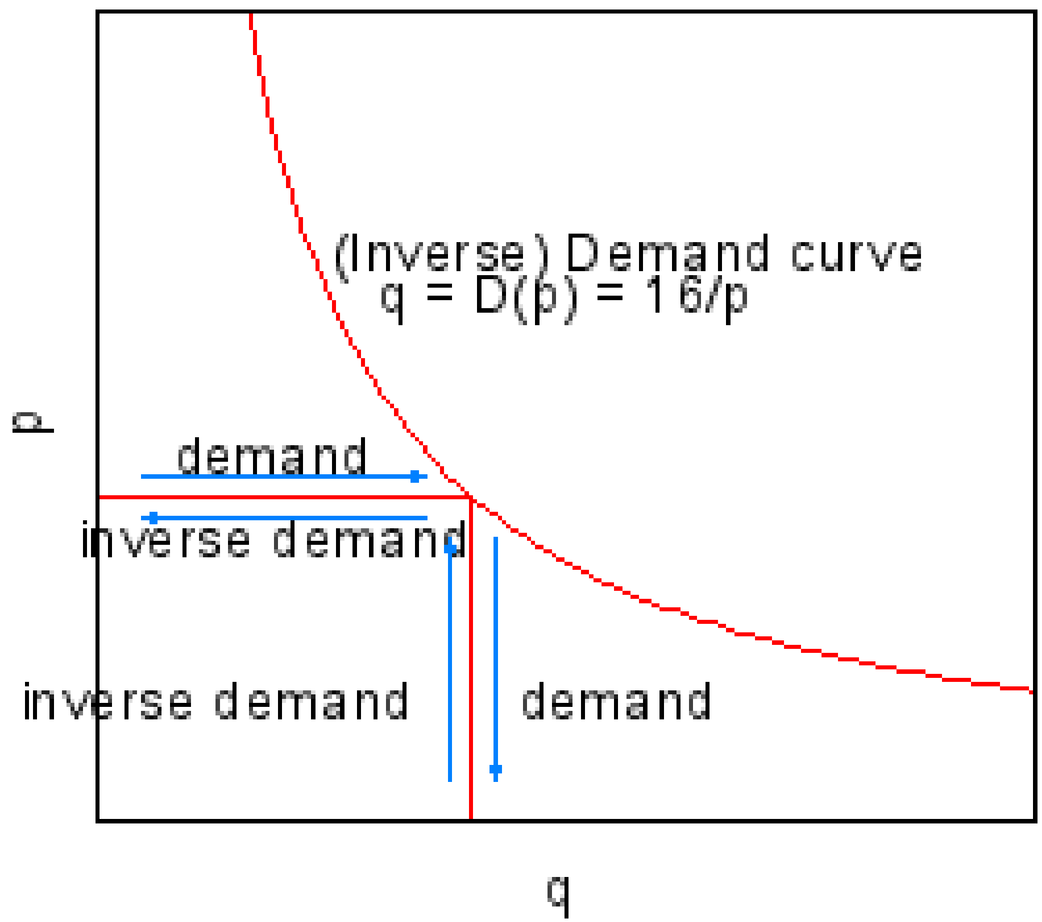
- Single-decision-maker, completely excludable exhaustible resource.
  - Social planner or monopoly; monopoly is simpler (expected discounted value of profit maximization)
  - Unlike intellectual property, consumers are completely excludable.
  - Monopoly is special: other producers are excludable. Government has the power to exclude, too.
- The monopoly decides how much to produce from the *stock* of the resource; the production provides a *flow* of benefits to consumers and is also deducted from the stock. This is similar to the saving decision in growth theory, where produced output (Y) can be used either as consumption ($(1-s)Y$) or savings (sY), and in the latter case it is *added* to the capital stock ($\dot{K}$).
  - We suppose production is costless for simplicity; the usual marginal analysis is possible, of course.
- Critical point: price of the good sold to consumer must equal price of the good saved as asset: they are perfect substitutes.

Consumer Behavior

- Consumer behavior: market demand curve.
  - Consumers do not store the good.
  - The demand curve will be the same for all market structures.
  - Examples: *linear* (constant-slope) and *constant-elasticity* demand curves.
- Assume the market demand function for the resource is constant over time. At each instant of time, the relation between price and total quantity demanded of the resource is the same.
  - After covering the basic theory, we will consider the adjustments that must take place in a growing economy.

Demand

- We denote the instantaneous or one-period *demand curve* by $q = D(p)$. Recall that we also use the *inverse demand curve* $p = D^{-1}(q)$, which has the same graph. The inverse demand curve interpretation is also called the *marginal willingness to pay* curve.



Dynamics and Demand

- With an exhaustible resource, price must eventually rise to choke off demand. Suppose current stock is S_0, and price is never (*i.e*, in no period) greater than $\bar{p}$. Then quantity demanded is never less than $\bar{q} \equiv D(\bar{p})$ since demand is downward sloping. For example, in the figure above, you could take $\bar{p} = 4$, implying $\bar{q} \equiv D(4) = 4$.
- Stock is exhausted no later than time $T \equiv S_0/\bar{q}$. But then at that point price must rise to make quantity demanded equal to 0, to maintain equilibrium.
- The time path price $= \bar{p}$ until stock runs out at time T, then jump to choke price, doesn't make sense. The inverse demand curve shows that the marginal customer at time T has much lower value than the marginal customer at time $T + 1$. Under a plan to exhaust the resource in time T, both a profit-making firm and a social planner want to reduce consumption in time T and increase it (to greater than zero) in time $T + 1$.
  - This applies to any pair of periods. Price rises gradually, forever.

Asset Pricing as Portfolio Choice

- The exhaustible resource is an asset, and is priced by comparing it to other assets, in particular, bonds. We consider whether the firm's future profitability is increased by selling more of the resource and buying more bonds, or by selling less of the resource and buying less bonds (*N.B.* marginal analysis).

- Suppose that at date 0 the market price of the resource is P_0. This is both the price as a commodity (sold to consumers) and as an asset (held for its future value).

- Suppose that the firm can buy or sell bonds with an interest rate of r. (That is, paying 1 yen for a bond today will yield a return of $1 + r$ yen tomorrow.)

Asset Pricing as Portfolio Choice, cont.

- Then the firm faces one of three situations about P_1, the next period price:
  - $P_0 > \frac{1}{1+r} P_1$. The firm should sell more of the resource and invest the cash in bonds. *I.e.*, the resource is overpriced today.
  - $P_0 = \frac{1}{1+r} P_1$. Neither the resource nor bonds increase in value faster; it doesn't matter which the firm holds. *I.e.*, the resource is correctly priced today.
  - $P_0 < \frac{1}{1+r} P_1$. The firm should sell less of the resource and not invest in bonds. *I.e.*, the resource is underpriced today.
- $P_0 > \frac{1}{1+r} P_1$ is derived from comparing two plans

  a. invest P_0 in the resource now, returning P_1 tomorrow;

  b. invest P_0 in bonds now, returning $(1+r)P_0$ tomorrow;

  the investments are the same, so we just compare the future values:

$$(1+r)P_0 > P_1.$$

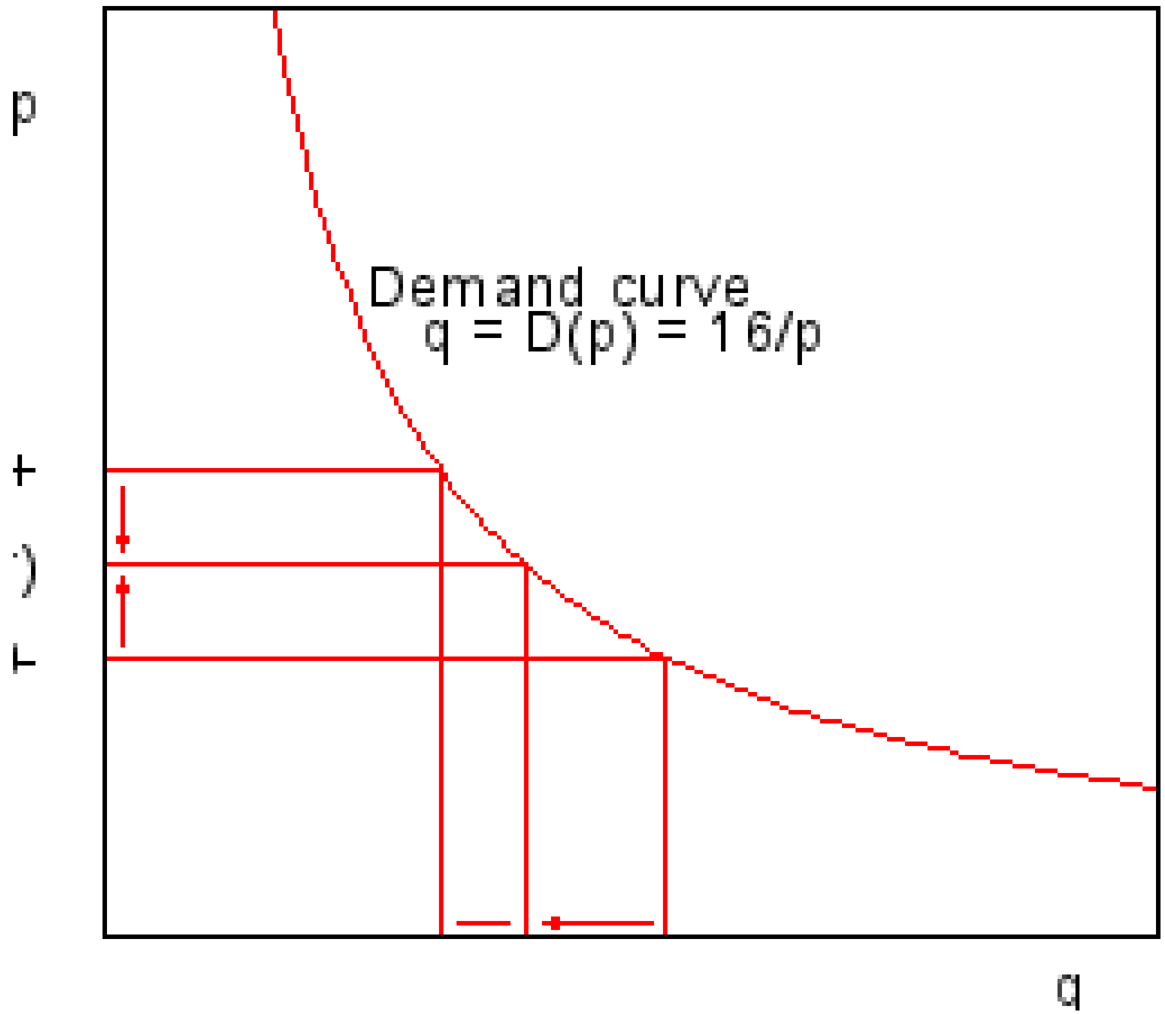
In present value terms ($\frac{1}{1+r}P_1$ is the present value of P_1) the comparison is as above

$$P_0 > \frac{1}{1+r}P_1.$$

- The argument above is called an *arbitrage* argument. *If* there is a sure way to make money by manipulating the financial markets, *then the market is not in equilibrium.* So, we have learned how to evaluate the price path by looking at the *financial market equilibrium.* This is common in dynamic studies.

Interaction of Investment Decision and Market Price

The firm is a monopolist, facing downward sloping demand. Its "investment decision" moves the terms of trade against it.



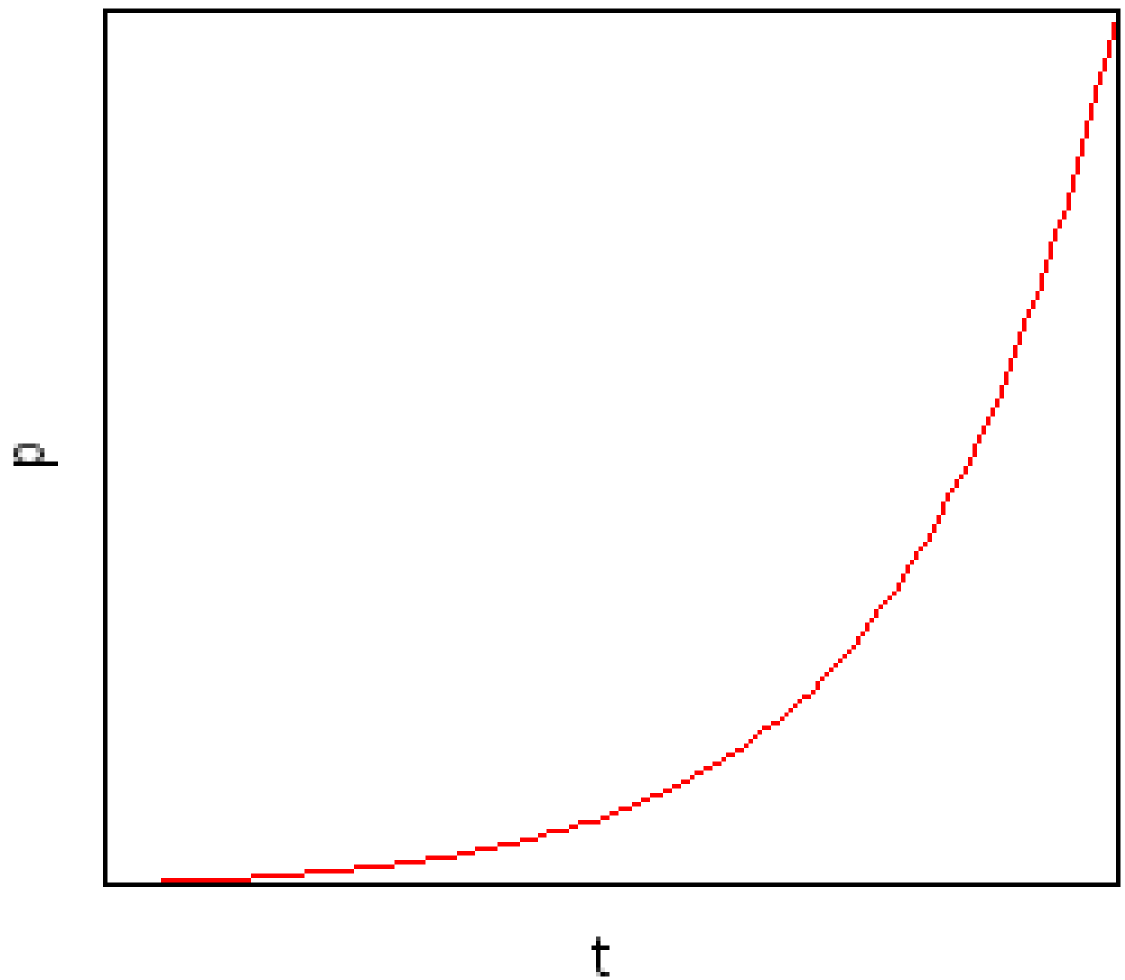
Dynamic Price Path

The firm must expect the price to rise over time according to the rule $P_{t+1} = (1+r)P_t$, *i.e.* $P_t = (1+r)^t P_0$.

Otherwise the firm will want to "play the market," but this also alters the market price of the exhaustible resource, returning the price to this path.

For given P_0, at time t the depletion is $D((1+r)^t P_0)$, and the stock is

$$S_T = S_0 - \sum_{t=0}^{T-1} D((1+r)^t P_0).$$



Initial Price P_0

- For given P_0, at time t the depletion is $D((1+r)^t P_0)$, and the stock is

$$S_T = S_0 - \sum_{t=0}^{T-1} D((1+r)^t P_0).$$

- Obviously price P_0 cannot be too low, or

$$S_0 < \sum_{t=0}^{\infty} D((1+r)^t P_0),$$

  and the stock is used up in finite time. This would drive price to infinity, higher than the path assumed ($(1+r)^t P_0$ is finite).

- What if price P_0 is too high?

$$S_0 - \sum_{t=0}^{\infty} D((1+r)^t P_0) > 0,$$

  and some of the stock is never used. This could happen for inelastic demand.

- Optimal initial price for monopolist depends on elasticity of demand.

Related Results

- If the choke price is not infinite (*e.g.*, with linear demand), in equilibrium the stock will be used up on the date when the price hits the choke price.
  - Interpretation: there is a perfect substitute whose price is the choke price.
  - So when price hits that level switch to the substitute.
- The social planner will surely use up all of the stock at infinity; this results in maximum benefit to society.
- If stocks are excludable, a competitive market in stocks will achieve the social optimum.
- If stocks are not excludable, the tragedy of the commons probably results in overexploitation, and possibly exhaustion in finite time.

The Hotelling Rule

- As usual, it is often convenient to solve a similar model in continuous time.
- By a usual kind of limiting argument applied the length of each period of time in the discrete arbitrage equation:

$$p(t+\delta) = (1+r)^{\delta} p(t),$$

we can compute a continuous price path according to the differential equation

$$\dot{p} = rp.$$

- This is called the *Hotelling Rule* (after the economist Harold Hotelling).

Note: I changed notation from capital P to small p because somehow small letters look better in differential equations. But it's the same variable, price.

Initial Condition and Equilibrium

- We have characterized the price path in terms of the no-arbitrage condition (the Hotelling Rule). But we haven't given the level of the price.
- First, integrate Hotelling's Rule to get $p_t = p_0 e^{rt}$ (again assume r constant over time for computation).
- Here, let us consider competitive supply (*e.g.*, in a case with many individual firms, each owning a single small oil field). Now let's try the consumption path where quantity supplied is equal to quantity demanded, at a price determined by inverse demand.
- Then the long-run resource balance condition gives $\int_0^\infty D_t(p_0 e^{rt})dt = S_0$, which is complicated but can be solved to get p_0^*.
- So we propose a price path of $p^*(t) = p_0^* e^{rt}$.

Verifying the Equilibrium

- Of course "supply = demand" strongly suggests market equilibrium, but we need to confirm that all agents are making optimal decisions.
- Is $p^*(t) = p_0^* e^{rt}$ an equilibrium? Yes:
  - The condition of consumers' optimum in each period is implied by the assumption that price is at the inverse demand of the quantity sold. Such consumers are *myopic* (near-sighted): they don't consider the future.
  - No firm can profit by selling more now: they will run out in finite time (because of the total resource balance, and by then the price will be just high enough to balance the interest rate.
  - The argument against selling less now is not the "mirror image"! The total resource is only a "hard" constraint against consuming more. With elastic supply, other firms will "fill in" the demand now, keeping price the same. Is this optimum for them? Yes, they are indifferent between earning now and later (when the *leader*, that is the firm that sells less now, sells its "excess reserve") because of Hotelling's Rule.

Uniqueness of Equilibrium

- Have we shown that there is *no* equilibrium that leaves some resource "left over at the end of time"?
- No! Because Hotelling's Rule makes them indifferent, the "other firms" *may* optimally choose to fill up the gap left by the leader now, and "give way" to the leader when it comes back in the future.
- However, there may be other equilibria. It may be that all of the firms decide to "follow the leader" and maintain a higher price. As long as forever after, the price follows Hotelling's Rule from the current price, no firm has an incentive to change its behavior and sell its reserve early, as long as it believes the other firms will *not* "give way" (and therefore the price will drop).

Is There a Proof of Multiple Equilibria?

- Can this argument be proved? The answer is that it depends on the elasticity of demand and the interest rate. If demand is very inelastic and the interest rate high, the profit to reducing supply now (and forcing price up) may be great enough to support a restriction on supply for ever.
- But if demand is elastic, the temptation to sell “a little more” now at (almost) no reduction in price will be very great, and the followers will eliminate the excess demand and return price to the level at which the stock is just exhausted “at the end of time.”
- Then that equilibrium is unique.
- A “bubble” is a price path that is not supported by fundamental value of the good or security. Normally bubbles are *irrational*, and eventually “burst” when it becomes clear to some agents that excess prices cannot be sustained by actual value. However, in this market, when the equilibrium is not unique, the non-competitive (high price) equilibria are what is called “rational bubbles”.

Backward Induction: An Example

- Here is a classic example of dynamic optimization from game theory: the stage game of the Chain Store Paradox. The setting is a large chain of stores (the *monopolist*) facing a new *entrant* with one store.
- After the entrant comes in, the monopolist has a choice of cutting price to chase him out, or sharing the market (here, at the monopoly price). See the table. Of course the monopolist will threaten to fight.
- There are *two* Nash equilibria in pure strategies (and none in mixed strategies). But is the deter/out equilibrium plausible?
- Not if the monopolist can decide on quantity *after* observing entry behavior, and the *entrant* plans ahead, thinking about the monopolist.

	deter	share
in	-1, -1	1, 1
out	2, 0	2, 0

Table 1: The Chain Store Paradox Game

Backward Induction: Adding Dynamics

- deter/out is implausible if the *entrant* plans ahead, thinking about the monopolist.
- This structure can be represented as a *tree.*

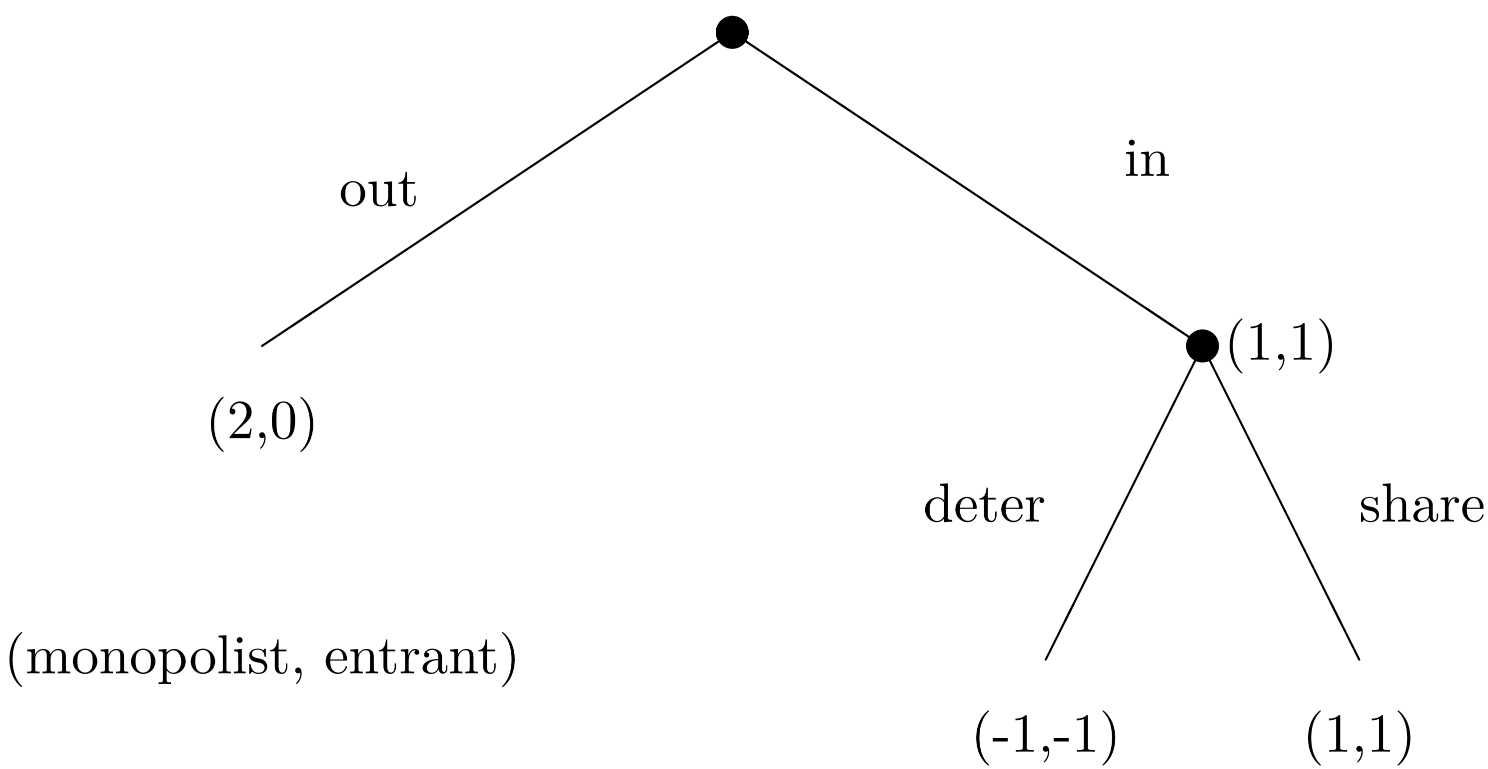


Figure 1: The Tree Form of the Chain Store Game

Backward Induction: “Perfect” Equilibrium

- The entrant thinks, if I enter, M’s $1 > -1$, so he will *share*.
- Therefore I should choose *in*.
- The monopolist, observing entry, does indeed want to *share*.

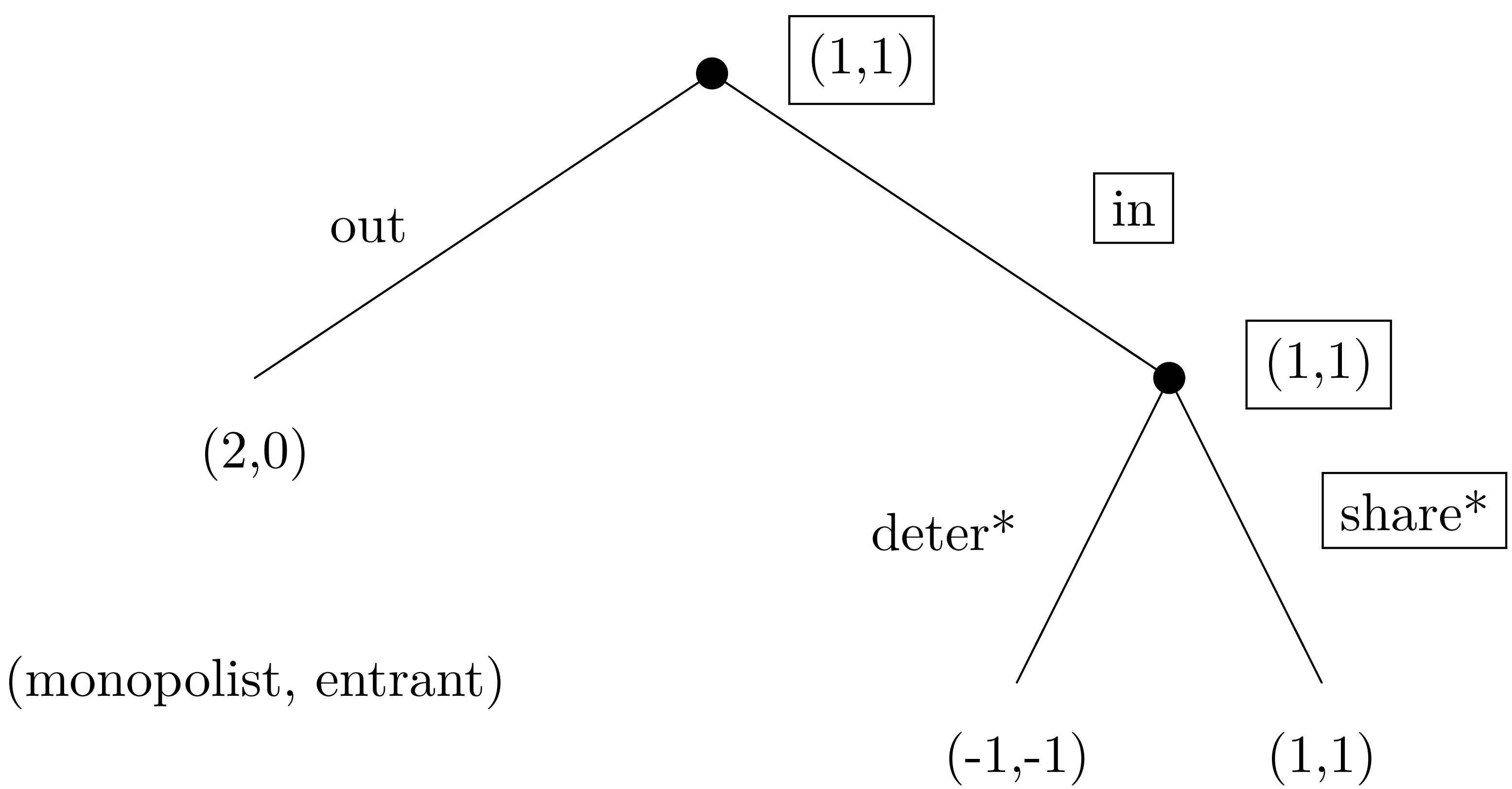


Figure 2: The Solution to the Chain Store Game

Backward Induction

- Backward induction is a method of solving dynamic problems.
- First, construct a *tree* to represent the dynamic structure.
  - Branches represent alternative behaviors.
  - The "chain-store paradox" game involves only agents' decisions, but we can also represent exogenous uncertainty as a "choice" by "nature" or "a god".
- Add payoffs to represent agent preferences.
- Start at the *end* (leaves) of the tree, solving each small subtree, then work "backward" to the root.

Economic Optimization

- Present discounted value
  - "Price" of profit in period t is $\left(\frac{1}{1+r}\right)^t$
  - Add up for all periods $\sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t \pi_t$
- Discounted utility
  - $\sum_{t=0}^{T} \delta_t U_t$
  - Usually $\delta_s < \delta_t$ for $s > t$
  - Often $\delta_t = \delta^t$
- If we propose a *plan* for the first T_0 periods, we can define a *value function* for the plan $x_0, x_1, \ldots, x_{T_0-1}$ (x_t is the decision at time t) as $\sum_{t=T_0}^{T} \delta_t U_t$.

Dynamic Decisions

Suppose you are planning for an expense far in advance. In 20 years you will need to have savings of \$100,000 (about 10,000,000 yen). There are many plans:

- Put \$100,000 in the bank today ($r = 0$).
- Put \$100,000 in the bank in 20 years.
- Put \$5,000 in the bank every year.
- put \$10,000 in the bank every other year.
- Put \$5,000 in the bank every other year, and in year 11 put in \$50,000.

In fact, there are an infinite number of ways. And the decision is very complex; for example, all of the above are probably impossible.

How can you make a plan for this?

Backward Induction

- It's hopeless to start out by picking a target savings this year. What you want to do this year will be affected by what you intend to do next year. And what happens if next year you realize that you really should have saved more this year? Won't you regret that?
  - So you have to make a whole plan for each of the next 20 years, and if you have any weak points, you must start over again.
- There is an important simplification that we can make because of the irreversibility of time. **Nothing you do in year 20 affects *feasibility* of actions in years 1–19.**
  - So year 20 actions depend only on your preferences and what you've done to that point. And you will *know* what you've done, it's history!

Backward Induction Example

We consider a smaller example, of only 4 years, needing \$20,000.

- For example, after year 3, you will have X_3 in savings, and you'll know exactly what X_3 is—it's in your account passbook. So your decision in year 4 is trivial: you must save $20000 - X_3$ to meet your goal.
- And after year 2, you'll have X_2 in savings. Your decision looks complicated: you must decide for year 3 and also year 4. But wait: decide your savings S_3, and that determines $X_3 = X_2 + S_3$. But then you automatically know what S_4 must be.
- Of course, the balance between year 3 and year 4 will be determined by the relative utility cost of saving (not consuming) in each year. But you can make this decision with only a limited amount of knowledge: X_2. That decision then forces the optimal decisions for all later periods (of course, there's only one).

Backward Induction Optimal Saving

- So, we can write
  - the optimal decision for savings in year 4: $S_4^*(X_3)$;
  - and for year 3: $S_3^*(X_2)$;
  - but wait, $X_3 = X_2 + S_3$, so we can write $S_4^*(X_2 + S_3^*(X_2))$.
  - Now both S_4^* and S_3^* depend on X_2.
- This process can be continued for period 2:
  - for year 2: $S_2^*(X_1)$;
  - for year 3: $S_3^*(X_1 + S_2^*(X_1))$;
  - for year 4: $S_4^*(X_1 + S_2^*(X_1) + S_3^*(X_1 + S_2^*(X_1)))$.

Backward Induction: Principle

- Written in algebra, it is quite complicated.
- However, when you solve for each period, you *assume* you have a solution for all later periods; there is actually only one "active" dependent variable (S_t) to solve for in each period, and one independent variable (the X_t). This is *relatively* simple.
- We can write a *formula* for each S_t, which we call $S_t^*(X_{t-1})$.
- This proves it's possible to solve but normally we don't solve by algebra. Instead, we use a *recursive* solution based on *value functions*.

First, let's finish the solution.

Backward Induction: Initial Conditions

- This could go forever, except for one thing: you must have an initial condition for X_0. For example, most college students can assume $X_0 = 0$. Since all the S_t^* are actually formulæ, $S_1^*(X_0)$ is a formula, and you can put the known value of X_0 in and compute it.
- Now you know $X_1 = X_0 + S_1$, and you can substitute into the formula $S_2^*(X_1)$, and so on. Working all the way back up the tree to $S_4^*(X_3)$, and you're done.
- This is very similar to the process of "Gaussian elimination" used in solving linear equations. The only difference is that the S_t^* functions are usually more complex.

Backward Induction: Utility and Value Functions

- The main problem left is the step where we decide the balance between one year and the (whole) future. This is done using a *value function*.
- Assume no discounting, an income of \$10,000 in each year, and utility of consumption in each period is $\ln C$. Then total utility is $\sum_{t=1}^{4} \ln C_t$.
- As before, $S_4^*(X_3) = 20000 - X_3$. It's forced.
- It turns out to be useful to figure out the maximum utility in terms of the previous amount of savings:

$$V_4^*(X_3) = \ln(10000 - S_4^*(X_3)) = \ln(X_3 - 10000)$$

- This is the **value function** of the savings after 3 years.

  We can assume assume a value function for "period 5" like $V_5(X_4) = -\infty$ for $X_4 < 20000$ and $V_5(X_4) = 0$ for $X_4 \geq 20000$, which is just a mathematical way of saying "forced to have 20000." This notation is useful because we can generalize to problems where there is *scrap value*, or there is a non-infinite penalty if $X_4 < 20000$.)

Backward Induction: Example and Value Functions

- Note that the value function gives a *future* value V_{t+1} in terms of a *current* variable X_t.
- Now when we come to year 3, for $S_3^*(X_2)$, we need to maximize $\sum_{t=1}^{4} \ln C_t$, but in fact this is $\sum_{t=1}^{4} \ln(10000 - S_t)$. Furthermore, we can no longer change our decisions about $S_1, \ldots, S_2$: they happened in the past.
- So we want to maximize $\ln(10000 - S_3) + \ln(10000 - S_4)$ subject to $X_2 + S_3 + S_4 = 20000$, which has *two* variables.
- However, we can take account of these by using the value function for the period 4 utility: $u(C_3) + V_4^*(X_3) = u(10000 - S_3) + V_4^*(S_3 + X_2)$ and now everything is in terms of predetermined variables (X_2) and a single decision S_3.

Backward Induction: Example and Value Functions, cont.

- So
$$V_3(X_2) = \max_{S_3} \ln(10000 - S_3) + \ln(X_2 + S_3 - 10000)$$
- The first order condition is
$$\frac{1}{10000 - S_3}(-1) + \frac{1}{X_2 + S_3 - 10000} = 0$$
which simplifies to $S_3 = 10000 - \frac{X_2}{2}$ and implies $S_4 = 10000 - \frac{X_2}{2}$.
- The *value function* is simply the maximized value of the remaining utility, and is given by substituting in the optimal savings:
$$V_3(X_2) = 2\ln(\frac{X_2}{2}).$$
- This is the **utility value of the savings** after 2 years.

Backward Induction: Example and Value Functions, cont.

- It is the next step that is the key. Instead of using the savings for periods 3 and 4 as tradeoffs to the savings for period 2, we use the value function!
- To determine S_2^*, we maximize $\ln(10000 - S_2) + V_3(X_2)$ subject to $X_1 + S_2 = X_2$, or more simply $\ln(10000 - S_2) + V_3(X_1 + S_2)$, which is

$$\ln(10000 - S_2) + 2\ln(\frac{X_1 + S_2}{2}).$$

- The first order condition is

$$0 = \frac{1}{10000 - S_2}(-1) + 2\frac{2}{X_1 + S_2}(\frac{1}{2})$$

or

$$X_1 + S_2 = 2(10000 - S_2)$$

- The solution to the first order condition is

$$S_2 = \frac{20000 - X_1}{3},$$

and the value function is

$$V_2(X_1) = 3\ln(\frac{X_1 + 10000}{3}).$$

Initial Conditions

- For period 1, we use its value function. To determine S_1^*, we maximize $\ln(10000 - S_1) + V_2(X_1)$ subject to $X_0 + S_1 = X_1$, or more simply $\ln(10000 - S_1) + V_2(X_0 + S_1)$, which is

$$\ln(10000 - S_1) + 3\ln(\frac{X_0 + 10000}{3}).$$

- The first order condition is

$$0 = \frac{1}{10000 - S_1}(-1) + 3\frac{3}{X_0 + S_1 + 10000}(\frac{1}{3})$$

- Restated

$$X_0 + S_1 + 10000 = 3(10000 - S_1)$$

- The solution to the first order condition is

$$S_1 = \frac{20000 - X_0}{4},$$

and the value function is

$$V_1(X_0) = 4\ln(\frac{X_0 + 20000}{4}).$$

Using the Initial Conditions

- The initial condition is $X_0 = 0$.
- This gives $S_1 = 5000$ and $X_1 = 5000$.
- Then $S_2 = \frac{20000 - X_1}{3} = 5000$ and $X_2 = 10000$.
- Then $S_3 = \frac{20000 - X_2}{2} = 5000$ and $X_3 = 15000$.
- Finally $S_4 = 20000 - X_3 = 5000$ and $X_4 = 20000$, of course.
- Note that $V_1(X_0) = \sum_{t=1}^{4} U(y_t - S_t)$. This is a general property: the optimal value of the value function for the whole problem is the optimal value. It's constructed that way.

General Formula

These require symmetry of y_t and U_t. They can be generalized to discounted utility easily.

- $S_t^*(X_{t-1}) = \frac{X_T - X_{t-1}}{T-(t-1)}$
- $V_t^*(X_{t-1}) = (T-(t-1))U(y - \frac{X_T - X_{t-1}}{T-(t-1)})$

Optimal control theory

- There are two basic approaches to dynamic optimization. The first is *optimal control theory*.
- Optimal control theory states the optimization problem mathematically as choice of a function of *time* from a set of such functions satisfying some constraints. The dynamic contraints are typically defined using a system of differential equations (or difference equations, when the problem is formulated in discrete time).
- The solution to the optimal control problem was first characterized by the Russian mathematician L. S. Pontryagin.
- Pontryagin's *maximum principle* is defined in terms of the derivatives of a function called the *Hamiltonian function* of the problem.

Dynamic programming

- The second approach to dynamic optimization is *dynamic programming*.
- Dynamic programming states the optimization problem mathematically as choice of a function of some *state* from a set of such functions. The special property of this function is that it gives the optimal value of solving the problem if we restart now.
- The solution to the dynamic programming problem was first characterized by the American mathematician Richard Bellman, using an equation called the *Bellman equation*.
  - In stochastic problems such as those encountered in finance (*e.g.*, the Black-Scholes model of option pricing) a more general form of the Bellman equation called the *Bellman-Hamilton-Jacobi* equation is used.
- Bellman's *principle of optimality* is defined in terms of the function of the state, which is called the *value function* of the problem.

Comparison of dynamic optimization methods

- Any given problem can be formulated as either an optimal control problem or a dynamic programming problem and solved. *Both methods must give the same optimal value, and usually the same solution.*
- The optimal control formulation is most useful for *open-loop* or "fire-and-forget" *solutions*, where the control is given as a function of time. These are common in physical situations (such as ballistics).
  - These require a very accurate formulation of the model. You have to be willing to commit to a specific plan for the control variable, and the solution method is not very helpful in understanding how to change the control if conditions change.
- Economics generally prefers to use dynamic programming (also called "recursive methods"), which gives *closed-loop solutions* where the control is given as a function of state, and thus can be adapted to unexpected changes in conditions.

Dynamic Optimization

- Simple backward induction based on comparison to next period.
- If we have discrete time and a terminal period, the value function for the last period is constant.
- Then we use the current utility plus (discounted) future value, optimize and get next-to-last period value function.
- This process continues until first period, and we're done.
- But what if we have infinite time or continuous time, or want a steady state?

Dynamic Programming in a Steady State

- "Backward induction" doesn't really make sense in continuous time or steady state, so we use more general term "dynamic programming".
- In discrete time the value function was

  $$V_t(X_{t-1}) = \max_{y_t}\{U(y_t) + \delta V_{t+1}(X_t)\},$$

  and in steady state this would be

  $$V(X_{t-1}) = \max_{y_t}\{U(y_t) + \delta V(X_t)\}.$$

- We can't find V without knowing V itself, it seems! This is a *fixed point problem.*
  - With an ordinary variable this is not hard. We have some function $f(x)$, and the fixed point problem is "solve $f(x) = x$ for x."
  - But for a *function*, this is hard.

Calculus of Variations

- General problem is "nonlinear programming in infinite dimensions."
- "Classical" approach is "calculus of variations".
  - Original use was computing things like shape of soap bubbles.
  - It uses methods like "Gateaux derivatives" and "Frechet derivatives" to derive first order conditions for the V, as we do in ordinary calculus.
- However, these methods are difficult and less general than modern *optimal control theory*.
- Advantage is direct relation to value function.

Derivatives with Respect to Functions

- A real-valued function of a function is a *functional*.
  - The definite integral of a function is a functional.
  - The value of a solution to a dynamic problem is a functional (typically expressed as a sum or integral).
- In finite dimensions, say optimizing the consumption of apples (q_a) and bananas (q_b), we take derivatives with respect to q_a, q_b, and λ of the Langrangean

$$L(q_a.q_b, \lambda) = U(q_a, q_b) + \lambda(y - p_a q_a - p_b q_b)$$

  E.g. for apples: $\frac{\partial L(q_a, q_b, \lambda)}{\partial q_a}$ and so on.

- But a function may have special constraint such as continuity.

Interpretation of the Value Function

- The value function is what is says, a way to evaluate the value of the state variable, *e.g.* a stock of an exhaustible resource.
  - We can compare it to the *market value* to evaluate the market performance: is $V(X) = PX$, where X is a state variable representing a stock of some good or productive input, and P is the market price of that good? If not, there is a *market failure* of some kind.

Interpretation of the Value Function's Derivative

- The derivative of the value function is comparable to the price of the state variable, which can be compared to *market prices*. Consider the derivative of V with respect to X, which is given in *units of value* (*e.g.*, money) per *unit of the variable*. Compare to the derivative of *market value*:

  $$\frac{d}{dX} PX = P.$$

  We saw a similar idea with the comparison of rate of interest to the derivative of H in the theory of renewable resources.

Optimal Control Theory

- The modern approach is *optimal control theory*.
- Developed from (literal) rocket science used to control missiles and warplanes as well as launching artificial satellites.
- More general than calculus of variations.
- Pontryagin Maximum Principle

Example: Optimal Growth

- We can easily transform the Solow model to an optimal control problem.
- We start from the characteristic differential equation

$$\dot{k} = sf(k) - (n+d)k$$

by recognizing that saving is no longer determined by a constant proportion, but by choosing consumption in each period c_t. So we substitute $f(k) - c_t$ for $sf(k)$

$$\dot{k} = f(k) - c_t - (n+d)k.$$

- The system is no longer autonomous, so let's rewrite to make time explicit

$$\dot{k}(t) = f(k(t)) - c(t) - (n+d)k(t),$$

giving the law of motion.

The Objective

- As usual, we want to maximize a per-person quantity.
- To trade off present versus future consumption, we use a concave utility function $u(c)$ with $u' > 0$ and $u'' < 0$, and a discounted sum (or really an integral, in continuous time)

$$\int_0^T u(c(t))e^{-\delta t}dt.$$

The Problem Stated

$$\begin{aligned}
\max_{c(t)} \quad & \int_0^T u(c(t))e^{-\delta t}dt \\
\text{s.t.} \quad & \dot{k}(t) = f(k(t)) - c(t) - (n+d)k(t) \\
& k(0) = k_0, k(t) \geq 0, c(t) \geq 0
\end{aligned}$$

The Hamiltonian Function

- The Hamiltonian formula according to theory is

$$H(k(t), c(t), t, p(t)) = q(t)u(c(t))e^{-\delta t} + p(t)(f(k(t) - (n + d)k(t) - c(t)).$$

- But the necessary condition requires $q(t) = \bar{q}$ for all t, and we can assume $\bar{q} > 0$ (otherwise utility has no effect on the solution!). Since as usual only relative "prices" matter, we can set $\bar{q} = 1$ as the numeraire, giving

$$H(k(t), c(t), t, p(t)) = u(c(t))e^{-\delta t} + p(t)(f(k(t) - (n + d)k(t) - c(t)).$$

- Note that the Hamiltonian is one equation at each instant of time. While that is a lot of equations, note that we have reduced this (in principle) to a set of static problems.

The Necessary Conditions

- The Hamiltonian formula is

$$H(k(t), c(t), t, p(t)) = u(c(t))e^{-\delta t} + p(t)(f(k(t)) - (n+d)k(t) - c(t))$$

(multipliers on the nonnegativity conditions for k and c are omitted).

- The necessary conditions for an optimum are

$$\begin{aligned} \dot{k}(t) &= \frac{\partial H}{\partial p} = f(k(t)) - (n+d)k(t) - c(t) \\ \dot{p}(t) &= -\frac{\partial H}{\partial k} = -p(t)(f'(k(t)) - (n+d)) \end{aligned}$$

and $c^*(t)$ maximizes $H(k(t), c, t, p(t))$ over c.

- The solution is

$$c^*(t) = -\frac{u'}{u''}(f'(k(t)) - (n+d+\delta)).$$

Interpretation of the Solution and Conditions

- By construction, the condition $\dot{k}(t) = \frac{\partial H}{\partial p}$ simply reproduces the law of motion. This is similar to the way $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$ reproduces the budget constraint in the Lagrangean formulation of the consumer's optimization: $\max_{x,\lambda} \mathcal{L} = u(x) + \lambda(y - px)$.

- The Hamiltonian is related to the value function. For this reason, it should be unsurprising that the condition $\dot{p}(t) = -\frac{\partial H}{\partial k}$ establishes a relationship between the "shadow price" p and the derivative of H with respect to k. (The negative sign is due to the fact that the Pontryagin Maximum Principle is from physics, and there naturally formulated as a *Minimum Principle*; but *maximizing* U is the same as *minimizing* $-U$, giving the opposite sign.)

- There is a *duality* relationship between the conditions for $\dot{p}$ and $\dot{k}$: each is characterized by the derivative with respect to the *other*.

More Interpretation

- The negative sign in $-\frac{u'}{u''}$ is due to $u' > 0$ and $u'' < 0$ as usual.
  - Note that u' and u'' are *time-varying*; they depend on $c^*(t)$.
  - We often use a *constant absolute risk aversion* (CARA) utility function in computing examples (even though there's no risk involved!), because then $-\frac{u'}{u''}$ is a constant.
- In a steady state, we get Solow's Golden Rule because

$$0 = -p(t)(f'(k(t)) - (n + d)).$$

- Note that the optimal path deviates only slightly from Solow's Golden Rule, by the introduction of δ in $n + d + \delta$.

Turnpike Theorems

- In many contexts (ballistics, various production processes, analysis of derivative securities, short-run growth planning), there is a specific state that should be achieved at the end of the planning period.
  - In the growth theory example, the "target" at the end of time is left free.
- Often the solution to such problems can be characterized in three steps:
  1. Consider the set of optimal "free" solutions, parameterized by the initial state.
  2. Find the optimal "free" solution whose path "goes through" the target state at the end of the planning period.
  3. Starting from the actual initial state, adjust the controls to go to the optimal free solution as fast as possible, then switch to the controls for that path.
- Such results are called "turnpike theorems" ("turnpike" is an old word for "highway").
- Turnpike theorems are *why* rockets burn all their fuel as quickly as possible, then coast to the destination.