Economic Dynamics

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Abstract

Today we discuss the role of difference and differential equations in modeling and analyzing economic dynamics.

Discrete dynamic systems

- The simplest dynamic models are self-contained, self-driving discrete systems like the doubling model of an epidemic.
- A *discrete dynamic system* is a sequence of values (scalar or vector) $\{y_t\}$ defined recursively. (The braces indicate that it is a sequence of y_τ for $\tau = 0, \ldots, t$.)
- *Recursive* means there is a *functional relationship* $y_{t+1} = f(\{y_t\})$ for $t = 0, 1, 2, \ldots$. These are also called *iterated function systems*.
- An alternative expression of a discrete dynamic system is a *difference equation* $\Delta y_{t+1} \equiv y_{t+1} - y_t = g(\{y_t\})$.
- We can define a recursive equation from a difference equation by $f(\{y_t\}) = g(\{y_t\}) + y_t$, and vice versa.
- Difference equations are closely related to differential equations, and are used in computational simulations.

Classifying Difference Equations

- We can take differences of differences:

$$\Delta^2 y_{t+2} = \Delta y_{t+2} - \Delta y_{t+1} = (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t)$$

  The exponent on Δ is called the *order* of the difference equation.

- A higher-order difference equation can be broken down into a system of first-order difference equations.

- If f is linear, the difference equation is called *linear*.

- f can depend on t. If it does not, the dynamic system is *autonomous*.

- We consider only the autonomous case where f is the same function of y for all t. This is more than a matter of simplifying computation and notation: it rarely makes sense to discuss *steady states* in non-autonomous systems.

- We classify recursive function systems the same way.

Difference Equations and Polynomials

- Consider a dynamic system where the position of a falling object is measured with respect to time. Suppose it started at rest at a height of c.
- Then its position at time t is $x_t = c - at^2$, where $a = \frac{1}{2}g \approx 4.9m/s^2$.
- Suppose the position is measured at one second intervals.
- The 1st difference is $\Delta x_t = x_t - x_{t-1} = (c - at^2) - (c - a(t-1)^2) = a - 2at$.
- The 2nd difference is $\Delta^2 x_t = \Delta x_t - \Delta x_{t-1} = (a - 2at) - (a - 2a(t-1)) = -2a$.
- Thus a second-order difference equation based on a quadratic time path is always a constant.
- Note that this analysis is based on the length of the interval analyzed being a constant.

A Recursive Equation Example

- Consider an infinite sequence of binary choices. We represent one alternative by "0" and the other by "1".
- An example sequence could be written "$z = .1101000110101\ldots$", suggesting the binary representation of real numbers between 0 and 1. In fact, this is the case, and there is a continuum of such sequences (a "big" infinity). Note: $z_0 = 1$, $z_1 = 1$, $z_2 = 0$, ...
- Consider the (non-autonomous!) dynamic system defined by a sequence of functions f_t such that $x_t = f_t(x_0, \ldots, x_{t-1})$ for $t = 1, 2, \ldots$.
  Evidently we can derive a new infinite sequence from any infinite sequence by applying this system to it. Call this sequence $f(z)$ (no subscripts!)
- Question: how big is the set $\{f(z) | z \in [0, 1)\}$?

The Surprising (?) Answer

- You might think that with all the flexibility of an infinite sequence of functions depending on everything in the past and a big infinity of z to choose from, the answer would be something like "a lot."
- In fact, the answer is **two**.
- The only part of z that affects the result sequence x is the *first* component z_0:

$$
\begin{aligned}
x_0 &= z_0 \\
x_1 &= f_1(x_0) \\
x_2 &= f_2(x_0, x_1) \\
x_3 &= f_3(x_0, x_1, x_2) \\
&\vdots
\end{aligned}
$$

The Surprising (?) Answer, *cont.*

- This sequence can be rewritten

$$
\begin{aligned}
x_0 &= z_0 \\
x_1 &= f_1(z_0) \\
x_2 &= f_2(z_0, f_1(z_0)) \\
x_3 &= f_3(z_0, f_1(z_0), f_2(z_0, f_1(z_0)) \\
&\vdots
\end{aligned}
$$

- $x = \{x_0, x_1, x_2, \ldots\}$ depends only on z_0, which must be either 0 or 1!

- In fact, the answer is **two**.

- From now on, we consider only sequences of scalars, and the case where the $\{y_{t-1}\}$ are ignored. The general case isn't harder in principle, but the computation and notation are tedious.

Discrete Processes

Some interesting processes really are discrete. Examples:

- The cobweb, Hicks, and Marshall adjustment processes can be considered as the result of (rather naive) behavior in a series of markets. For example, agricultural markets from one year to the next. (It is the inelasticity of demand and supply for food that made Marshall decide to use a model where quantity *adjusts* and price *responds to the quantity adjustment*.)
- Human beings cannot be continuously active. They must sleep, best with consistent daily sleep cycle. Thus it makes sense to measure human activity per day.
- Most activities consist of a sequence of tasks. The tasks may take variable amount of clock time to complete, but it often makes sense to count "time" in terms of tasks completed rather than watching the clock.
- Newton's method for solving equations.

Discrete Simulations

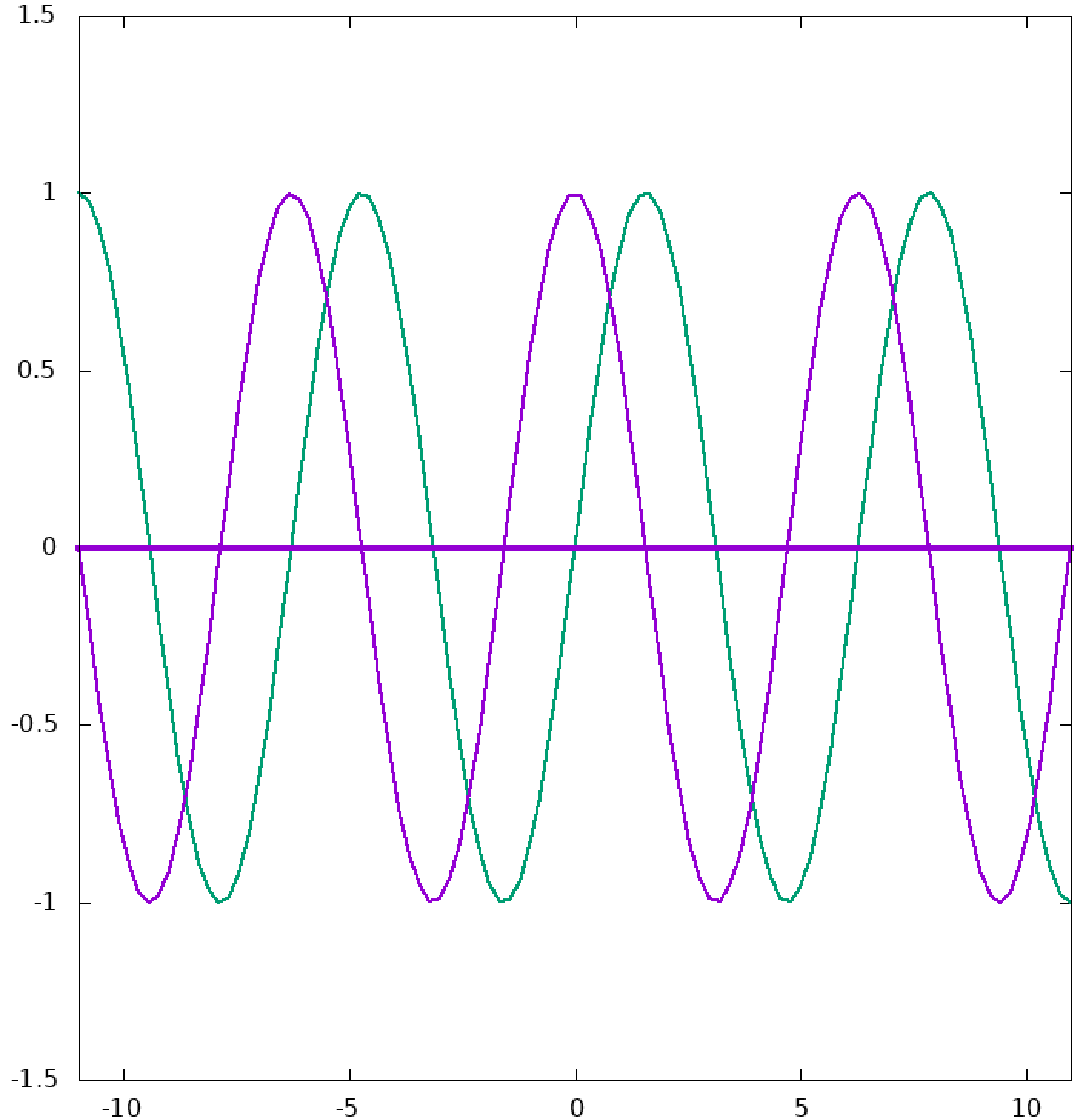
But one of the most important uses of discrete processes is *simulation* of continuous processes.

- You've probably used the *Artisoc* or *Netlogo* software in jisshuu to model crowd behavior in buildings or in disaster response.
- Weather simulations: global warming, general local weather, and specific phenomena like typhoons or El Niño.
- Numerical computation of processes such as price adjustment in a market.

Differential equations

- A *differential equation* is a mathematical expression of a constraint involving certain variable quantities and their derivatives.
  - A differential equation is often called a *law of motion*, but they arise in other contexts, such as determining the shape of a chain hanging from two points.
  - A set of differential equations relating a specific set of variables is called a *system* of differential equations.
- Differential equations can be graphed in many ways:
  time series, phase diagrams, vector fields, and
  many others.

Time series: sin x *vs.* cos x

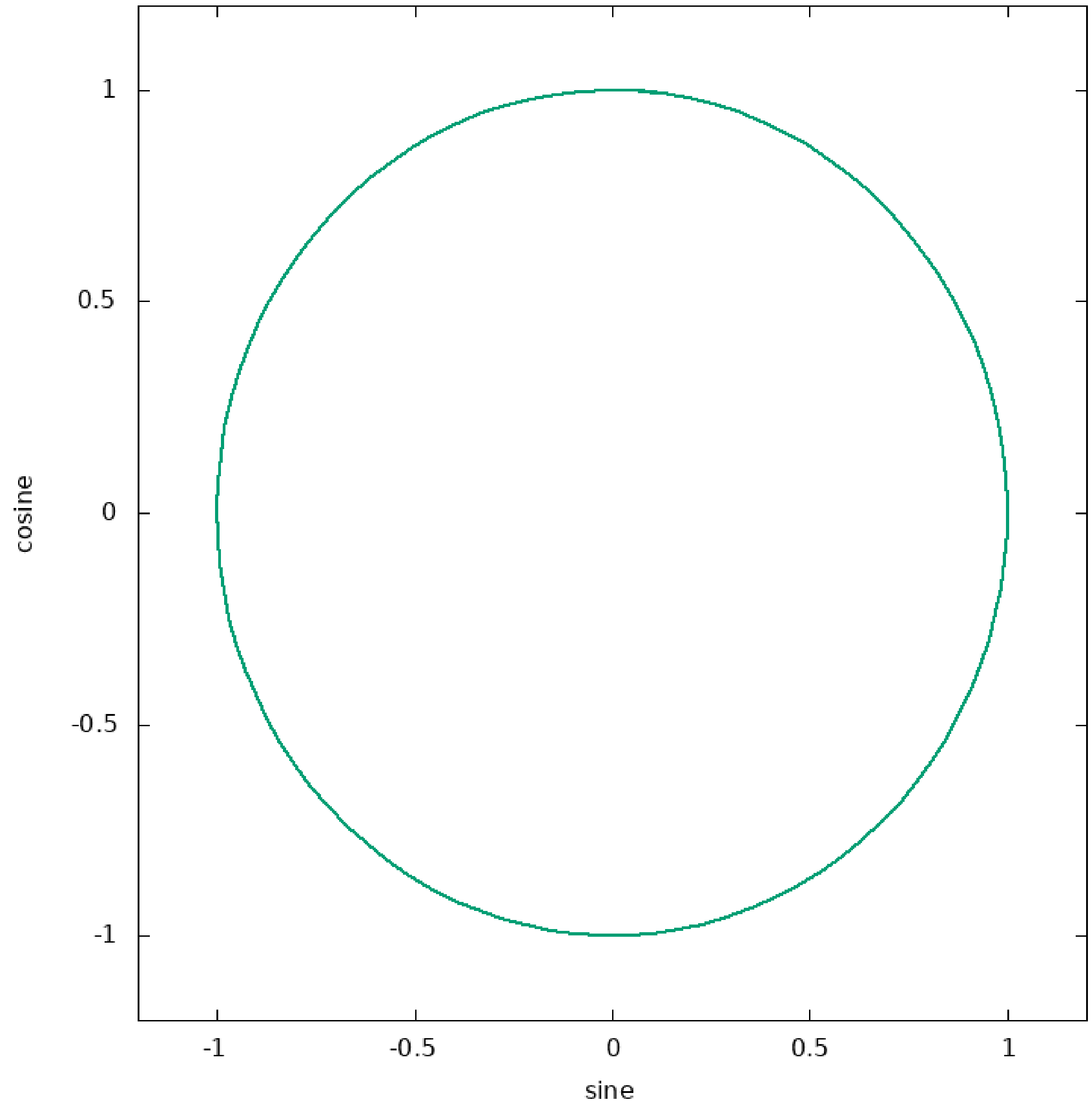


Sine and cosine chase each other (high and low) across the graph.

Phase diagrams

- In economics, the most useful graph is the *phase diagram*, which graphs the combinations of values of two variables that occur simultaneously. Phase diagrams do not show *when* the combinations occur, only that they do at some point.
  - The *parameter* (in dynamics, the *time variable*) is implicit.
  - Because of *continuity* the phase diagram shows the order of values. Thus the graph is often called an *orbit*.
  - It doesn't show direction of time flow.
- For example, you saw the graph of $\sin x$ and $\cos x$ against x. This is not a phase diagram. However, you can see that they are "chasing" each other. The exact relationship is displayed on a phase diagram of $\sin x$ against $\cos x$, and it is the *circle* $\sin^2 x + \cos^2 x = 1$.

Phase diagram: sin x *vs.* cos x



Sine and cosine chase each other (high and low) around the circle.

Qualitative properties of differential equations

- In economics, we often do not have very precise information about differential equations we use. *E.g.*, $dx/dt = f(x, t)$, with some marginal conditions on f.
- Even if we do, we may not be able to solve to get an explicit function $x(t)$.
- In looking at *economic growth theory*, we will focus on one set of qualitative properties: those involving the steady state, x such that $\dot{x} = 0$. *Market stability analysis* also revolves around steady states, with the special property that they should correspond to the supply-demand equilibrium.

Classifying differential equations

- If all of the unknown quantities are functions of one of the quantities, all of the derivatives may be reduced to ordinary derivatives, and the equation is called an *ordinary differential equation*.
  - The single quantity is called the *parameter*. In dynamics, the parameter is interpreted as *time*.
  - Otherwise, partial derivatives are involved, and the equation is called a *partial differential equation*.
  - If a system of differential equations contains any partial differential equations, it is classed as a system of partial differential equations.
- The *order* of a system of differential equations is the order of the highest derivative involved in the system.
  - We are primarily interested in *first-order differential equations* of the form $\frac{dy}{dx} = f(x, y)$.
  - Differential equations of higher order may be reduced to systems of differential equations.

Example: free fall

- According the Newton's Law of Gravity, when an object is allowed to fall freely to the ground, its acceleration toward the ground is constant.
- We denote the height of the object at any time t by $h(t)$.
- The *velocity* (speed *and direction*) of the object is the *first derivative* of height, denoted $\frac{dh}{dt}$, $h'(t)$, or $\dot{h}(t)$.
- The *acceleration* of the object is the first derivative of velocity, or the second derivative of height, denoted $\frac{d^2h}{dt^2}$, $h''(t)$, or $\ddot{h}(t)$.
- Since it is constant, we have a *second-order* differential equation $h''(t) = g$.

The general solution for free fall

- We *solve* the differential equation by finding an equation with no derivatives in it.
- This is "simply" a process of integrating the equation as in basic calculus. Each integration lowers the order of the differential equation by one, and when the order reaches zero, we're done.

$$\begin{aligned} h''(t) &= g \\ \int h''(t)\,dt &= \int g\,dt \\ h'(t) &= gt + C_1 \\ \int h'(t)\,dt &= \int gt + C_1\,dt \\ h(t) &= \frac{1}{2}gt^2 + C_1 t + C_2 \end{aligned}$$

- where C_1 and C_2 are *constants of integration.*

Specific solutions for free fall

- g, C_1, and C_2 are arbitrary constants, and we cannot use them to compute the height of the object numerically until we know their values.
- The values are determined from other facts about the problem. The "standard" problem specifies that
  - g is known from previous experimental measurements.
  - Time is measured in seconds since the object was set free.
  - The object was at rest at time 0, so $0 = h'(0) = g \cdot 0 + C_1 = C_1$.
  - The height of the object at time 0 was measured to be h_0, so $h_0 = h(t) = g \cdot 0^2 + C_2 = C_2$.

Example: Soap bubbles

- Why are soap bubbles spherical?
- The mathematical model of a bubble is based on a system of partial differential equations which characterize equality of air pressure inside and outside of the bubble.
- These differential equations have a *spatial parameter*, *i.e.*, the position of each point on the bubble. So differential equations need not be based on a time parameter (though in economic dynamics they are).
- The soap bubble model is completed by describing it as an optimization problem which minimizes surface area subject to the system of differential equations.

Unconstrained population growth

- We assume some species (bacteria, plants, animals, firms in an industry) with
  - specific resource requirements for reproduction,
  - an unlimited supply of those resources, and
  - no predators.
- Then the increase of the species population may be expected to be proportional to population with coefficient *fertility rate*:

$$\frac{dP}{dt} = \alpha P.$$

- Collecting variables gives $\frac{dP}{P} = \alpha\, dt$, then integrating gives $\ln P = \alpha t + C$, and finally exponentiating

$$P(t) = P(0)e^{\alpha t},$$

where $P(0) = e^C$ is the population when $t = 0$.

Constrained population growth

- Assumptions are as in unconstrained growth, except that due to resource limitation or increased predation, the death *rate* increases proportionally to population.
- Here the differential equation is

$$\frac{dP}{dt} = B - D = \beta P - (\delta P)P = P(\beta - \delta P).$$

- The solution is

$$P(t) = \frac{\beta}{\delta + [\frac{\beta}{P(0)} - \delta]e^{-\beta t}}.$$

- It is not easy to derive this solution directly,
  and we don't need to do that.
  Verifying that it *is* a solution is assigned as homework.

Approximating $y' = ax - by$ and $y(0) = a/b$

We can find an approximate specific solution to $y' = ax - by$ and $y(0) = a/b$ where $a > 0$ and $b > 0$ using the following procedure:

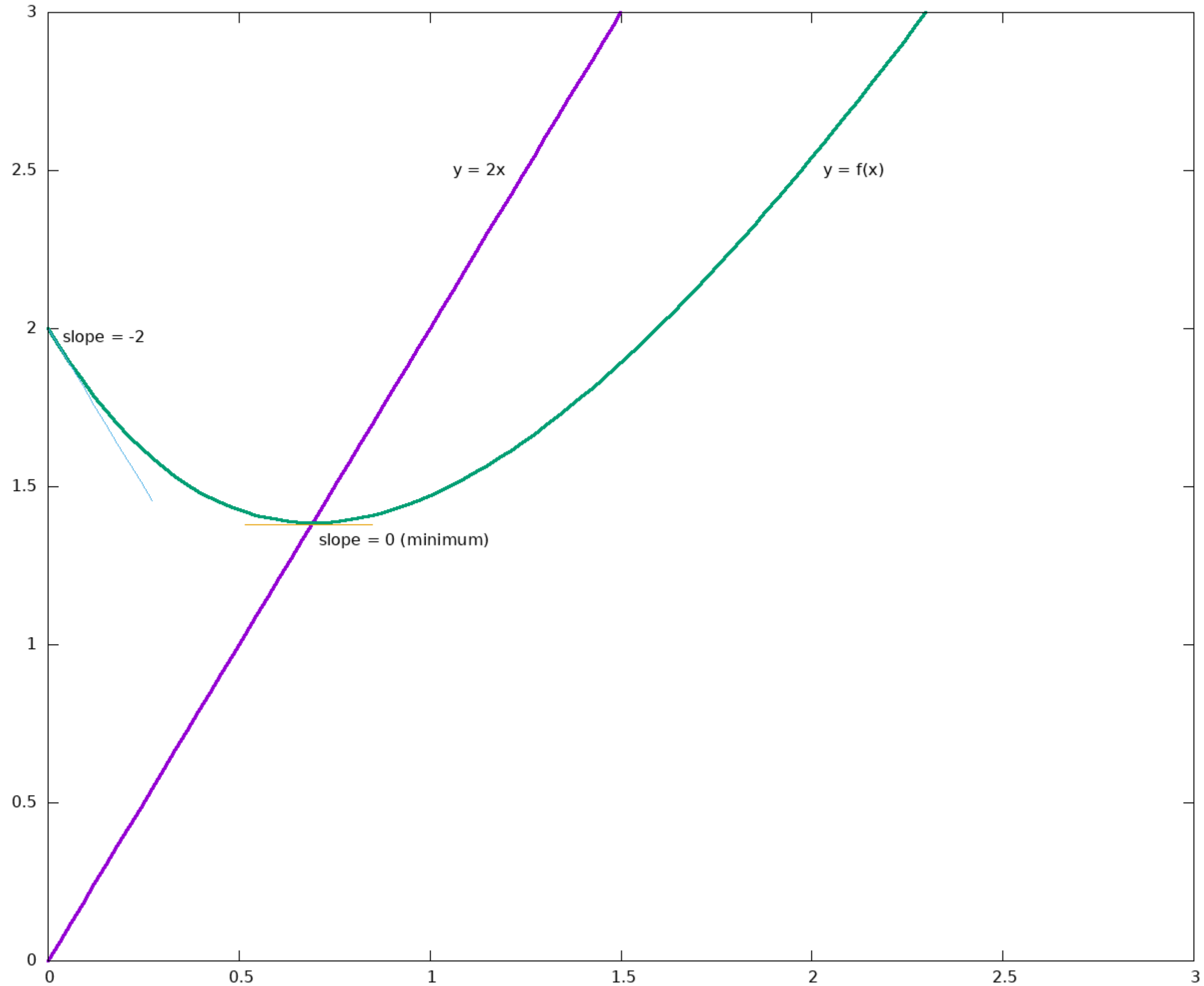
- Plot the point $(0, a/b)$ on the graph.
- Evaluate $\frac{dy}{dx}$ at $(0, a/b)$, getting $-a$, so the specific solution f (*i.e.*, $y = f(x)$ for all x) is downward-sloping at that point.
- Assume that f has a minimum. Then at that point $\frac{dy}{dx} = 0$, so $y = \frac{ax}{b}$. (We know the derivative exists because f is the solution to a differential equation!)
  - We need to check that the intersection of f and $y = \frac{ax}{b}$ is a minimum, not an inflection point, of f.

    $$\frac{d^2y}{dx^2} = a - b\frac{dy}{dx} = a > 0,$$

    because $y = \frac{ax}{b}$ is defined so that $\frac{dy}{dx} = 0$, and f is convex, so this is a minimum.
- The curve must look something like the next graph.

Approximating $y' = ax - by$ and $y(0) = a/b$

We use the differential equation directly, plus the boundary condition ($y(0) = a/b$) to compute some of the points on the curve $y(x)$, and its shape.



Isoclines

- "Cline" is a Greek word meaning slope. "Iso" means "equal" in the same language. Thus, "isocline" means "something" has the same slope in different places. In fact, an *isocline* is the set of all points where that "thing" has a given slope.

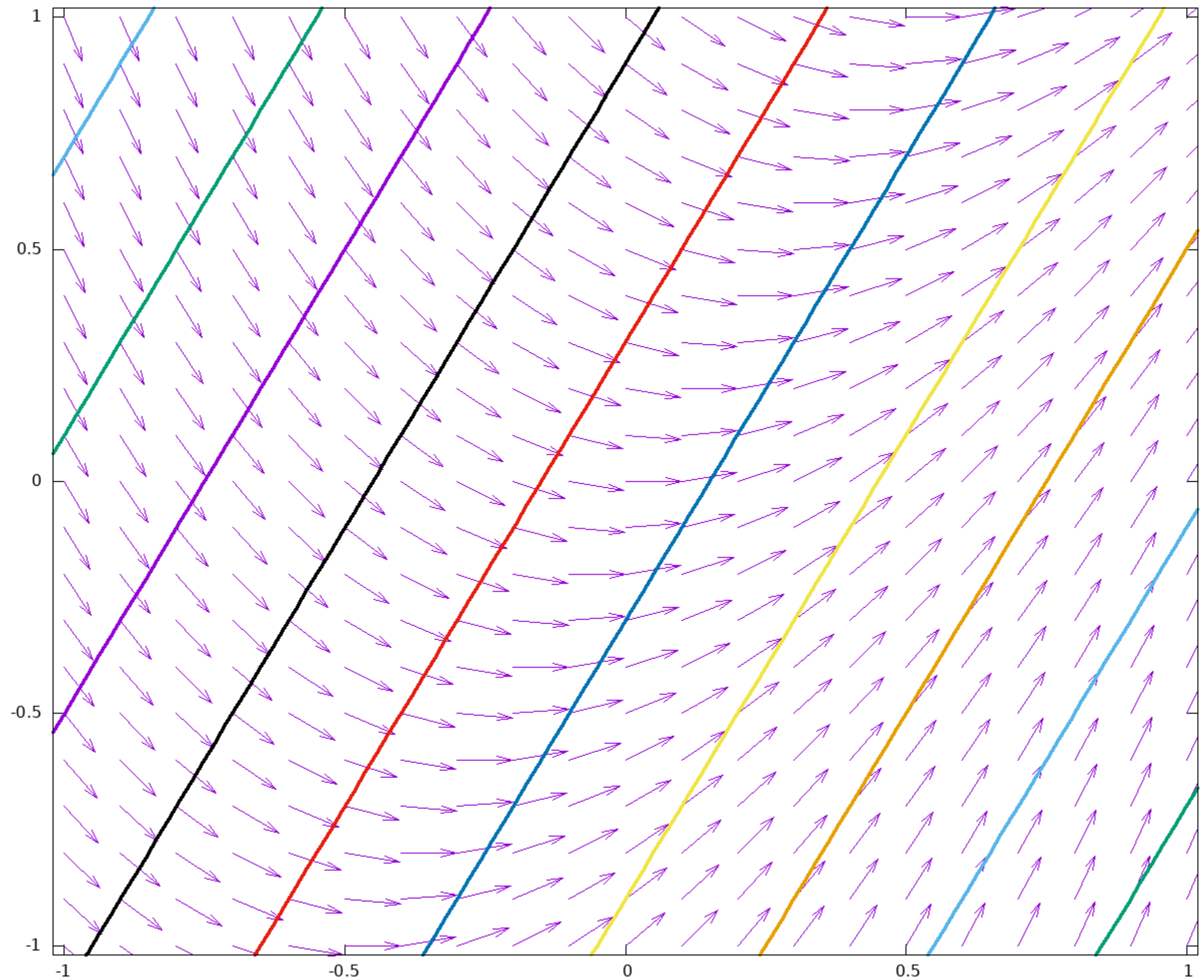
- In our specific solution, by definition, f has the same slope m for all points satisfying $m = \frac{dy}{dx} = ax - by$. So the *general* solution has isoclines with the parametric equation $y = \frac{ax-m}{b}$. m can be any number.

- Isoclines need not be linear.

- The isocline for $\frac{dy}{dx} = 0$ is special, because any steady state must be contained in that isocline.

Isoclines of $y' = 2x - y$

Each arrow shows y' at the point of its tail. The arrows that cross each isocline have the same slope along the isocline.

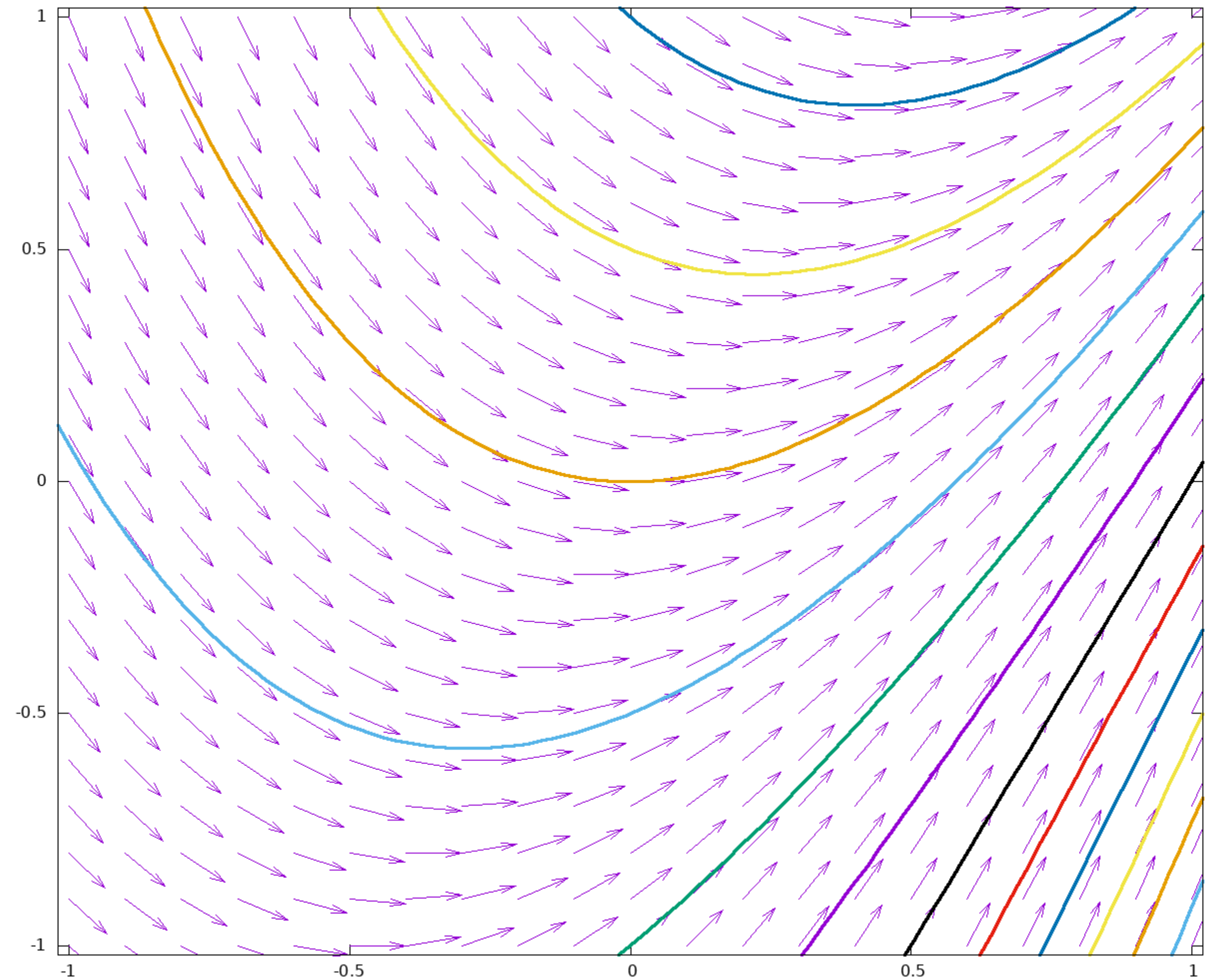


Direction fields

- An alternative representation is to attach the slope implied by $\frac{dy}{dx}$ to each point.
- This is call a *direction field*.
- If we start at some point and "connect the arrows" head to tail, we get a curve called an *integral curve*. This is the graph of a specific solution.

Direction field of $y' = 2x - y$

The curves are different trajectories depending on the starting point. See how they track the arrows showing y' at each point.



Other properties

- In the image of the direction field, it seems that there's a "main stream" crossing from the 3rd quadrant through the 4th quadrant and then going up and to the right forever in the 1st quadrant. *This is correct.*
- Consider the isocline going through $(0,0)$. The isocline has slope 2, but the direction field's value is 0. So the particular solution going through $(0,0)$ will "move off" the isocline.
- Since our isoclines are linear with slope a/b, let's consider the condition $\frac{dy}{dx} = \frac{a}{b}$. Then $\frac{a}{b} = ax - by$, and $y = \frac{a}{b}x - \frac{a}{b^2}$. Thus this condition identifies an isocline!
- Of course this is harder to do for nonlinear $\frac{dy}{dx}$, but when possible it is very useful.

Solving differential equations computationally

- In one sense, a differential equation always has a solution. That is, the fundamental theorem of calculus says that for an integrable function $f(t)$, $\int_\ell^u f(t)dt = F(u) - F(\ell)$, where $f = \frac{dF}{dt}$, and F is continuous and differentiable.

- In practice, we can always compute a time path for $f(t)$ by simulation (picking a value for $F(\ell)$, then setting
$F(\ell + (n+1)\delta) = F(\ell + n\delta) + \delta f(\ell + n\delta))$
for δ "sufficiently small").

- However, this is not generally very useful in economic theory (though it is frequently used for examples and actual simulations).

Characterizing solutions

- Because of *resource limitations*, “explosive” growth by individual entities cannot continue indefinitely. From the point of view of individuals in an economy, there should be some stability.
- History shows that individuals *can* usefully predict (near) future conditions by assuming they won’t be (much) different from current conditions, so there is a degree of stability.
- For these reasons, not all differential equations are useful models of economic phenomena.
- We also want to characterize the solutions in terms of
  - “where they settle down” (existence of steady states)
  - “how fast they settle down,” (stability of and speed of convergence to steady state), and
  - “optimal control” (where the direction and speed of change can be controlled by policy).