Economic Dynamics

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Abstract

We ended our study of Solow's growth model, which is *unconstrained* and *purely dynamic*. Usually considered a *macroeconomic* model, it can be applied to single industries and other contexts when "valued added" is the sole concern.

Next is *exhaustible resources*, where the problem is to allocate a *bounded* stock of a resource to use over time. We start with the *renewable resource case* for its analytical similarity to Solow's growth model.

Note: This homework is due on the day of the midterm examination. *Plan your work accordingly.*

Optimization in Dynamic Models

- Solow's basic *growth model* is entirely dynamic. No optimization is present at all.
- The "Golden Rule" is determined by using steady state to reduce the growth model to a single period (although it is repeated indefinitely), and then treating s as a choice variable in a *static* optimization.
- Now we look at *exhaustible resources*, both
  - *renewable resources* (concentrating on the *fishery*), and then
  - *pure exhaustible resources* (such as oil),

  as first examples where economic considerations (optimization and equilibrium) enter.

Oil: A Pure *Exhaustible* Resource

- Oil exists as a stock, and is used up, *i.e.*, "exhausted." Once the stock is exhausted, there will never be any more.
  - This is an approximation: after 20 million years or so, new stocks will form in the ocean floors. But we use up oil *much* faster than that.
- Oil is *storable*. This implies that the rental price (price of consumption) is equal to the asset price. This is true of all inventories of course, but in the economics of the firm we usually equate assets to (generic) capital, where this is not true. (Compare: is the price of a rental car equal to purchase?)
- Resources like oil that must be used up are called *pure exhaustible resources*.
- Since oil cannot be recovered once used up, several conceptual issues arise:
  - The model should have an infinite horizon.
  - Steady states are impossible.
  - Intergenerational considerations are important.

Pure Exhaustible Resources and Renewable Resources

- Exhaustible resources come in two varieties: *pure exhaustible resources* like oil, and *renewable resources* like fish. *Renewable resources* are "self-renewing" in that if left alone they will grow back to the original stock. However, like pure exhaustible resources (and unlike capital) there is an upper bound on the stock.
- Biological resources like fish are also similar to pure exhaustible resources in that once exhausted, there will never be any more (Jurassic Park excluded).
- Exhaustible resources are necessarily *storable*. Such resources have the property that the rental price (price of consumption) is equal to the asset price. (Compare: is the price of a rental car equal to purchase?)
- You might think that a pure exhaustible resource is just a special case of renewable resource, with the *natural rate of increase* set to zero. In some ways this is true, but each type has its own natural mode of analysis.

Summary: Exhaustible Resource Definition

- An *exhaustible resource* is a good where the stock is bounded and the natural rate of increase has an upper bound, but can be consumed arbitrarily quickly. If completely depleted, no more will ever exist.
  - A *pure exhaustible resource* cannot increase at all. Once any portion is consumed, that quantity is gone forever.
  - A *renewable resource* is an economic good whose stock automatically renews itself at some rate, but this rate (and the stock itself) has some upper bound.
- Renewable resources would better called *self-renewing* resources, but *renewable* is traditional usage and can't really be changed now.

Examples of Exhaustible Resources

- A *pure exhaustible resource* can only be used up; it cannot increase. Oil (or any other mineral resource) is a good, and very important, example of a *pure* exhaustible resource.
- A *renewable resource* is one which has a positive *rate of natural increase*, for at least some level of the stock. Tuna fish (or any other wild biological resource) is an example.
- Solar energy is often called a "renewable energy source." That is reasonable English usage, but solar energy is not a *renewable resource* in the sense used here.
  - The Sun will always be there again tomorrow (no exhaustion).
  - It provides more energy than we can imagine using in the foreseeable future if only we could capture it (effectively unbounded).
- A habitat's "ability to absorb pollutants" may be analyzed as a renewable resource. Consider how Lake Erie (U.S.) or Lake Kasumigaura managed to recover from heavy pollution.

Introduction to Renewable Resources

- "Renewable resource" is an odd term because "renew" is a transitive verb.
  - It suggests investment (*i.e.*, the resource is physical capital).
- These resources might better be called *self-renewing* resources (at least, they "self-renew" if left alone).

Renewable Resources

- Dasgupta and Heal (*Economic Theory and Exhaustible Resources*, 1979) describe these as "resources that are at the same time self-renewable and *in principle* exhaustible."
- Not mineral resources, called "exhaustible," whose stock cannot be increased, although more efficient technology (*e.g.*, deeper oil wells) and discoveries of new stocks may arise.
- Not durable commodities. If the entire stock of automobiles were destroyed, we could still build more and replace them. This might depend on available exhaustible resources, but if the resources were available, the automobiles could be replaced.
- Populations of biological creatures, and agricultural land, are examples. Self-cleansing of polluted water is an example.

Populations

- We borrow terms and ideas from biology, but the exact definitions used often differ.
- A *population* is a group of individuals of the same kind which reproduce themselves, perhaps only in the context of the group. The *population size* is the number of individuals. (In biology, it is often the *total mass*, because as resources become strained individual size decreases. We abstract from this issue.)
- In nature populations tend to increase over time. This increase should depend on
  - the period of time θ over which increase is measured
  - the current population Z_t
  - other factors ζ_t

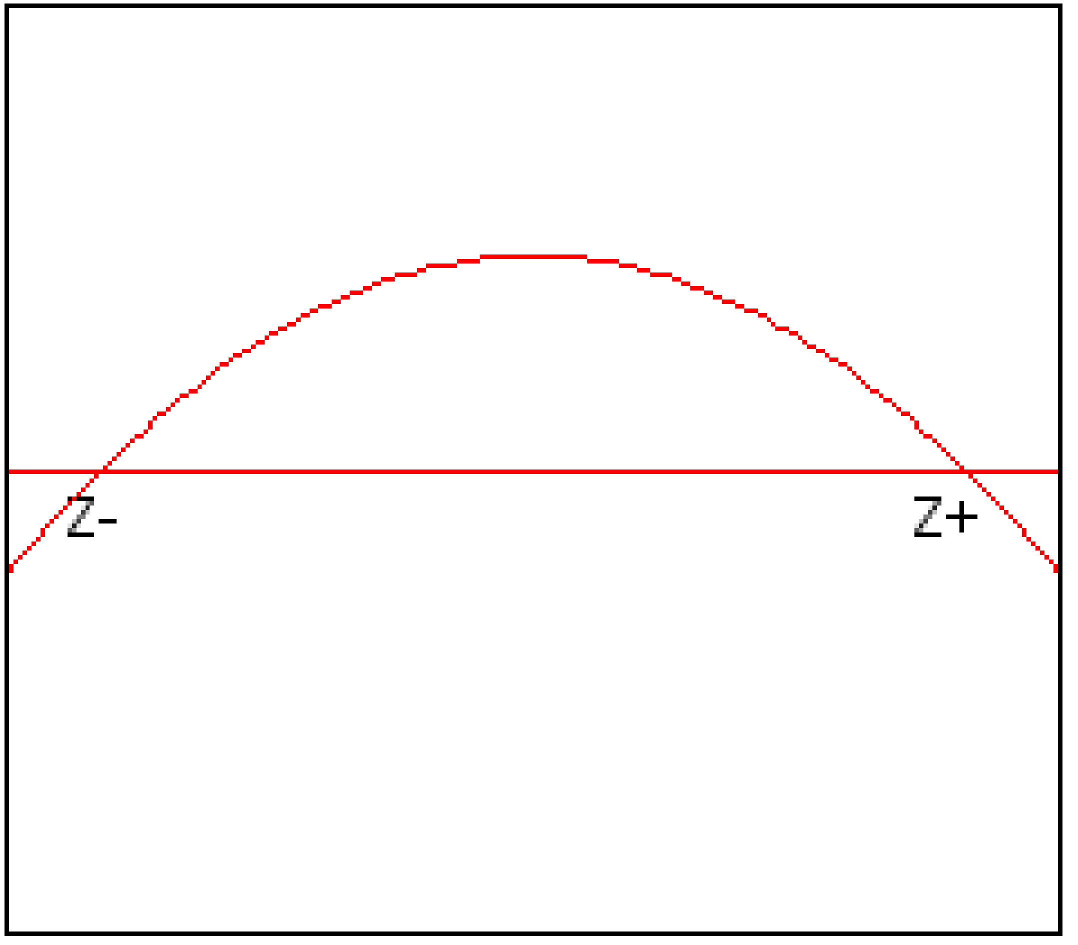
  giving $Z_{t+\theta} = G(Z_t, \zeta_t, \theta)$.

Typical Laws of Increase

- We are typically interested in the *natural rate of increase* of a population.
- If ζ is constant, then we may drop its notation. We typically assume that θ enters multiplicatively, and we write $G(Z_t, \theta) = Z_t + H(Z_t)\theta$.
- With continuous time, we rewrite as the difference, divide by θ, and take the limit as $\theta \to 0$:
$$\lim_{\theta \to 0} \frac{Z_{t+\theta} - Z_t}{\theta} = \dot{Z}_t = H(Z_t).$$
This *defines* H.
- Example: $H(Z) \equiv 0$ is a constant population size. This means renewable resources include exhaustible resources as a special case.
- Example: with abundant resources, $H(Z) = \lambda Z$, giving exponential growth.
- Example: often, $H(Z)$ is a bell curve.

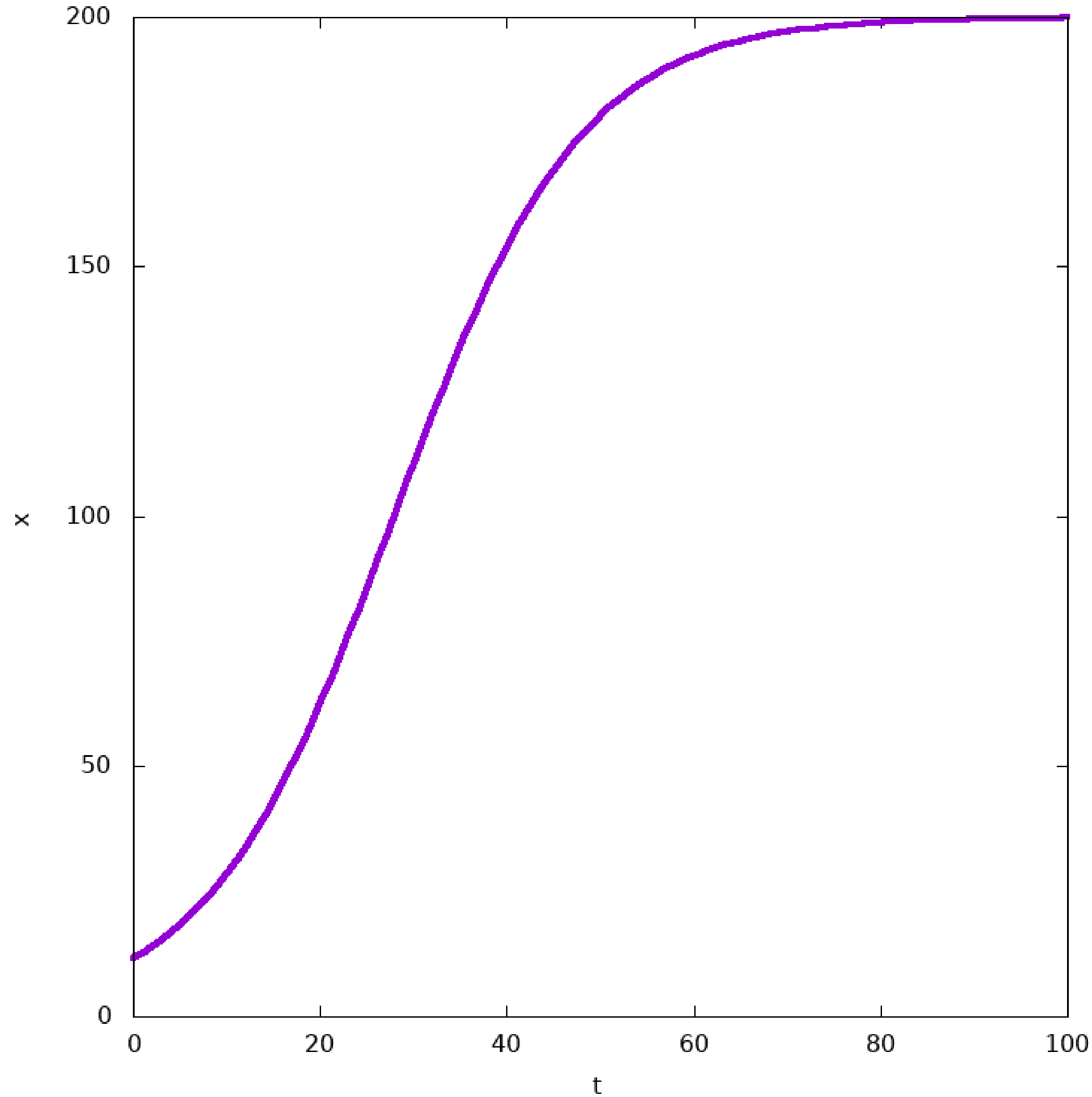
Bell-Shaped Curves

- $H''(Z) < 0$ for all $Z \geq 0$.
- $0 \leq \underline{Z} < \bar{Z}$ such that $H(\underline{Z}) = H(\bar{Z}) = 0$.
- Unique $\hat{Z}$ where $H'(\hat{Z}) = 0$ and $H(\hat{Z}) > 0$.



Logistic Growth

The "S-shaped" logistic growth curve is a typical time path for a bell-shaped natural rate of increase function (in logistic growth, $H(Z) = aZ(1 - bZ)$, an inverted parabola.



Comment on Extreme Cases

- We described pure exhaustible resources as a boundary case of renewable resources.
- Although we can use the same specifications to describe them, the mathematical analysis of the typical case often doesn't hold for the boundary case.
- Example: linear functions are a special case of convex functions. But a linear cost function doesn't allow us to determine the scale of firms in the industry—it doesn't matter if there is one firm or many, the cost structure is the same. However, strictly convex costs (decreasing returns to scale) will have a unique scale for each firm in equilibrium.

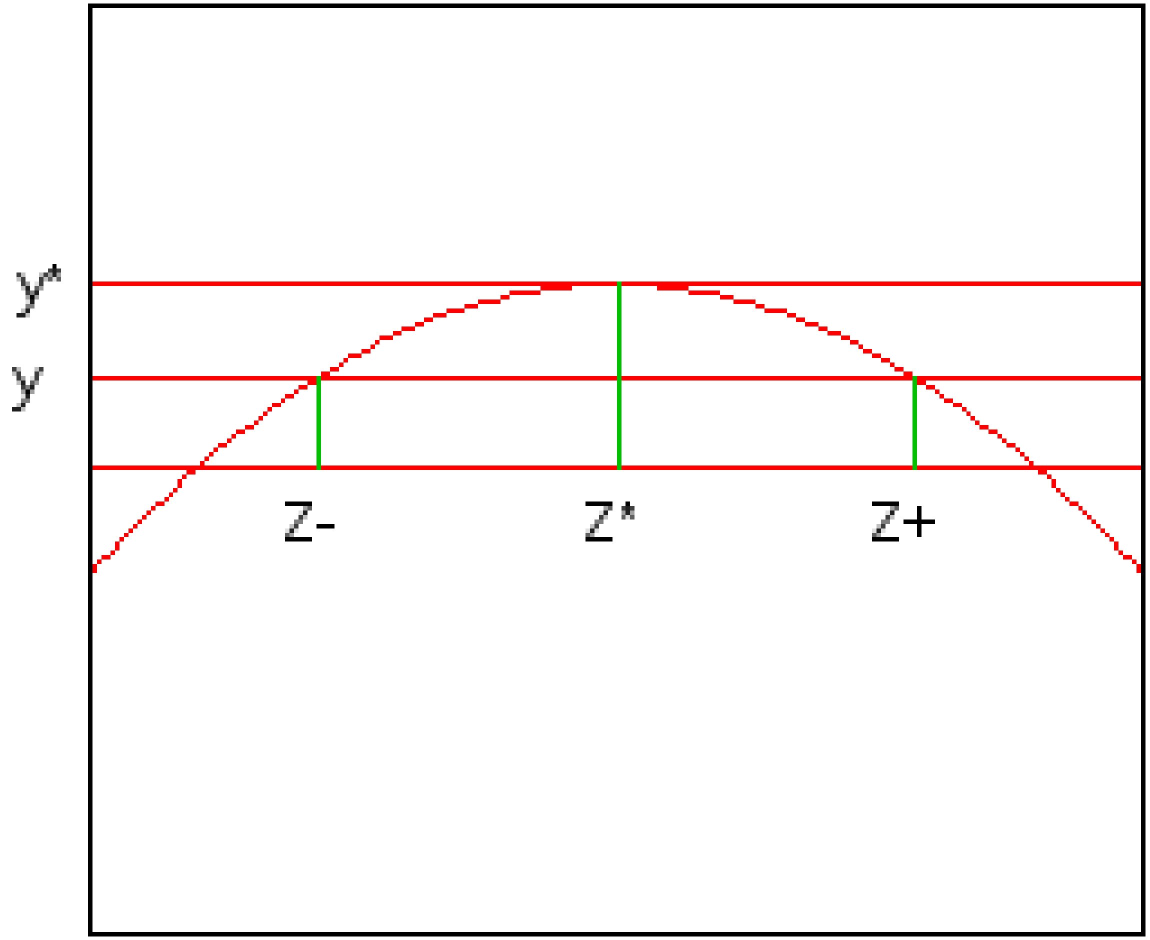
Harvesting the Fishery

- How does the dynamic of the fishery change when we add large-scale fishing to the model?
- In economics, we simply remove fish from the population. In a differential equation model, we represent this by the rate of harvest at each instant of time, and write
$$\dot{Z}_t = H(Z_t) - y_t.$$
- In biology, we would also consider
  - Collateral damage to predators and prey of the commercially valuable fish. For example, tuna nets often catch (and drown) dolphins.
  - Pollution from fishing activity (*e.g.*, spilled oil and gasoline) might kill the target fish.
  - Fishing activity might interfere with reproduction.

  Our simple model ignores these factors but they could be added.

Constant Harvest Rate

Consider the stationary policy $y_t = y \leq y^* = \max H$ for all t. Note that there are two steady states at Z^- (unstable) and Z^+ (stable), unless $y = y^*$, when the unique steady state at Z^* is *unstable.*



The Case $y = y^*$

- When $y = y^*$ there is a steady state at Z^* which is considered *unstable*.
- Looking closely, we see that although for $Z < Z^*$ where $H(Z) < 0$ and Z diverges from Z^*, for $Z > Z^*$ again $H(Z) < 0$, and Z *converges to* Z^*.
- Couldn't we say Z^* is "half-stable" or "stable from the right"? Formally, yes, but as a model it doesn't make sense. We *don't know* why we deviate from Z^*, so it could be up or down. Eventually it will be down, and divergence will occur. If it reaches $\underline{Z}$, extinction occurs (divergence is *permanent*).
- Conceptually (*i.e.*, in modeling) "stable" means "always converges." This approach is *useful* in modeling because we may assume that once the steady state is reached, the system will be "approximately in steady state" *forever*.

Harvest Policy in a Feedback Loop

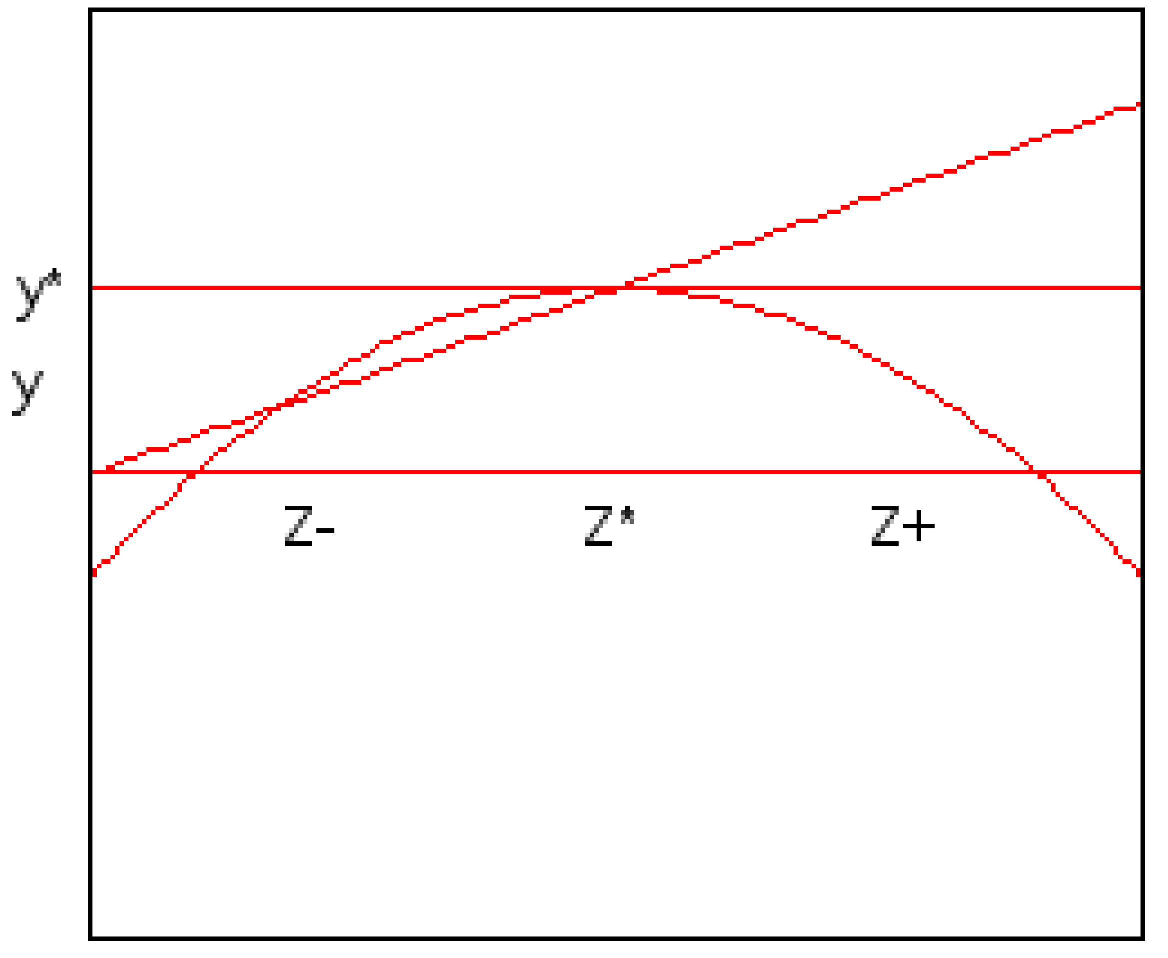
- The fact that the maximum harvest steady state is unstable means that a *laissez faire* policy toward the fishery is very *risky*. A policy which ignores the population and simply takes the optimal catch will eventually lead to extinction.

- Use the policy $y_t = y^* + \xi(Z_t - Z^*)$ to achieve a stable maximum harvest steady state.
  - This policy is called a *negative feedback loop*. The path of Z it induces is a *closed-loop solution*. (It's *negative* because in Z phase space we have

    $$\dot{Z} = H(Z) - h(y) = H(Z) - (y^* + \xi(Z - Z^*))$$

    so the direct coefficient on Z is $-\xi$.)
  - The constant policy is called an *open-loop solution* (generally, any policy which is not a function of Z).
  - Note that, though simple to state, this is not necessarily an easy policy to implement: how do you count the tuna in the sea? If many fisherman are involved, how do you share the cost of the census?

Closed-Loop Harvesting



Use the *negative feedback policy* $y_t = y^* + \xi(Z_t - Z^*)$. Note that the unique steady state is *stable*.

When Do Steady-State Policies Make Sense?

- This is an *economic* (not "purely dynamic" as with Solow) issue. Need an objective to optimize (until now, implicitly "maximize steady state y").
- Need an infinite planning horizon, else you want to use up the population "before the world ends."
- Entry must be controlled (else "tragedy of the commons" occurs as new entrants try to "grab their share" by consuming earlier than future entrants do).
- How about "maximum sustainable" harvest y^* (with a closed-loop policy)?
  – No discounting, or time preference outweighs higher future consumption.
  – With discounting, a steady-state optimal policy exists on the left side of the bell—open loop is unstable.

Dynamic Optimization in the Fishery

- Consider a monopolist owning a whole fishery (*e.g.*, in a lake), selling fish at a fixed competitive price, with no variable cost of catching fish.
  - Unreasonable, but in fact a technical assumption: the basic result (maximum sustainable catch is not dynamically optimal) holds for "usual" cost functions.
  - Profit maximization is straightforward, but the same kind of analysis works for social planner too.
- There is a fixed cost: interest on a large loan taken to buy the fishery. Interest is charged on the remaining principal. Also suppose the fishery was at its stable steady state population when purchased.
- In dynamics, a *turnpike theorem* states that when a system is "off the ideal course," the optimal policy is to move to the ideal course as quickly as possible.

The Case $Z > Z^*$

- As long as $Z > Z^*$, the monopolist will want to catch fish as fast as he can. The argument is straightforward, there is no tradeoff:
- He could take the same fish now or in the future: catching now reduces the loan, and interest costs in the next period. Else he'll have to pay higher interest for one period, and he has to reduce his loan someday.
- Catching now doesn't reduce the possible catch in the long run, because the population is currently growing slower than its maximum rate—catching fish actually increases the steady state yield.

The Case $Z \leq Z^*$

- At Z^*, catching a tiny amount more than $H(Z^*)$ decreases H by "almost nothing," while the decrease in principal owed means that interest paid decreases by a small amount—but more than nothing.
  - Thus it is profitable to have a stock $Z < Z^*$ with steady state yield less than maximum because cost (interest payment) decreases more.
  - Define $\tilde{Z}$ to be the optimal steady state stock accounting for interest payments.
- Computing $\tilde{Z}$ case requires a precise notion of "marginal productivity" to compare to the marginal cost of interest. We approach this via the concept of arbitrage, the return to waiting, and the own rate of return.

Arbitrage

- The most important concept in finance is *arbitrage.* That is, exploiting a configuration of markets that allows you to buy a good at a low price in one market and immediately sell it in another, guaranteeing a profit.
- Buying low, selling high among separated markets for real commodities involves transportation and storage, *i.e.*, production. These aren't *pure* arbitrage; *pure arbitrage* really applies only to financial markets.
- There can be differences between prices in geographically separated markets (such as for the dollar or for gold) that are open at overlapping times (*e.g.*, London and New York). They are very quickly erased by arbitrage.
- Arbitrage is most useful for understanding pricing of real assets by comparison to specific interest rates. (Geographically separated financial markets can be treated by considering them to be one global market whose trading floor moves occasionally.)
- We will determine $\tilde{Z}$ by arbitrage of $H'(Z)$ against the interest rate r.

Rates of Return, Discounting, and Interest

- A central concept in finance and dynamic economics is the return to waiting, usually measured by an interest rate.
- An *interest rate* (strictly defined) is a rate of compensation for use of a sum of money for a given period of time.
- (Market) interest rates equilibrate the time preferences of consumers and the productivity of firms.
- A consumer's time preference is expressed in terms of a *discount rate*, which has the same units as an interest rate.
- Productivity of investment by firms is expressed in terms of the *rate of return*, which also is formally the same as an interest rate.

Own Rate of Return

- We can measure the value of saving/investing, *i.e.*, by not harvesting the population, in a natural way even if there is no market price or interest rate.
- Measure the value in terms of the good itself (rather than money).
- Suppose the good is "self-reproducing": given a stock of the good Z, after an interval of θ there will be $G(Z, \theta)$, for some function G. (It could be that $G_Z(Z, \theta) < 0$ for all Z – an "iceberg," or $G_Z(Z, \theta) > 0$ – a bank account, or any shape – living population.)
- Suppose we hold one more unit of the good. Then the extra good in the following period, over and above the unit we hold, is approximately $G_Z(Z, \theta)$. (*I.e.*, the total amount extra in the next period is $1 + G_Z(Z, \theta)$.)
- Not $1 + G(Z, \theta)$, which is the return to the entire population Z. We're interested in the amount of increase due to the extra investment: $\frac{G(Z+\delta,\theta)-G(Z,\theta)}{\delta}$, which when taken to the limit as $\delta \to 0$ is of course $G_Z(Z, \theta)$.

A Caution

- Regarding own rate of return, it is sometimes claimed that the population would change, and that causes difference between G_Z and G. Not true: the point here is the usual difference between *average* and *marginal* which is central in modern economics.
- We want to compare an increment of δ to investment, then we have $G(Z+\delta,\theta)$ next period, the net gain is $G(Z+\delta,\theta)-G(Z,\theta)$.
- Dividing by δ and taking δ to zero gives the definition of the derivative of G, *i.e.*, G_Z.

The $\tilde{Z} < Z \leq Z^*$ Case

- At $Z = Z^*$, the argument for $Z > Z^*$ applies "at the margin": the interest rate is strictly greater than zero, the principal has to be paid sometime, and the maximum steady state isn't quite decreasing.
- For $Z < Z^*$, if Z is close enough to Z^*, the maximum steady state catch decreases very little, and it can be made smaller than the interest payment at the specified rate (a constant).
  - Thus it makes sense to decrease steady state catch in return for decreasing principal, and so reducing interest paid.
- When is Z "close enough" for interest to be larger than the lost steady state harvest? That is determined by the own rate of return: by the usual "marginal argument" the benefit of reduced interest is just offset by the cost of lost steady state harvest when $H'(\tilde{Z}) = r$ (the interest rate).

Homework 7: November 2, 2018

Note: Homework 7 is due on *Friday*, as usual.

Consider a renewable resource with the natural rate of increase

$$H(Z) = 1.5Z(1 - 0.05Z).$$

1. What is the differential equation governing the change in population if the rate of catch is y?
2. Solve for the stable and unstable steady states Z^+ and Z^- in terms of y. Be careful about the admissible values of y!
3. Considering the expression you derived in part 2, interpret the meaning of that expression for *inadmissible* values of y, and explain why they're called "inadmissible".
4. Solve for the maximum sustainable (constant) catch y^*, and the population Z^* that supports it.
5. Pick a "reasonable" value for ξ in the closed-loop policy $y = y^* + \xi(Z - Z^*)$
6. Explain why you consider your value of ξ from part 5 "reasonable".

Steady States are Impossible with Pure Exhaustible Resources

- Actually, there is an infinite set of possible steady states, but they are all uninteresting, except for one, and that one is impractical.
- Steady states where the state variable is the stock involve the resource staying constant, and therefore zero consumption. This is uninteresting (and clearly not economic equilibrium: somebody will "cheat" by consuming!)
- As with growth theory, we can consider a state variable which is per capita stock. But since positive consumption requires decreasing stock, to maintain a steady state with positive consumption implies decreasing population.
  - This requires matching two goals precisely: the willingness of people to accept restrictions on family size and the preferred level of resource consumption. That seems hard to do in practice, and almost certainly not through a decentralized mechanism like the market.
- This implies we need a utility function (criterion for trading consumption in one period against consumption in another).

Intergenerational Considerations

- Since people live a finite time, it is nearly certain that their preferences for consumption during life will differ from their preferences for consumption later, even if they care about their children and future generations.
  - They may value them less, or
  - Care about their (differing) utility rather than specific consumption, or
  - Be imperfectly informed about the future.
- But the current generation's actual choices change the *constraints* for future generations in a way the future generations cannot affect the current generation.
- This is different from growth theory where you can always return to the "golden rule" steady state.
- For simplicity, we will ignore issues of intergenerational equity, but they are extremely important in practice (consider Japan right now!)

Exhaustible Resources Compared to Growth Theory: Assumptions

- We consider resources which exist in finite amount, and necessarily are depleted by consumption.
- Contrast growth theory based on factors which are produced and accumulated without bound: capital, technology.
- It's not enough to just change the sign. There is a lower bound of *zero* for exhaustible resources; they cannot keep decreasing at a given rate forever.

Exhaustible Resources Compared to Growth Theory: Modeling

- Because of depletion, we cannot use steady state analysis.
- Steady state analysis, although based on a *dynamic model*, allows us to think about the economics in the same way as we do with equilibrium.
- Without a "good" steady state to aim at (*e.g.*, Solow's Golden Rule), we *must* make explicit intertemporal tradeoffs, *i.e.*, dynamic economic analysis. It's not possible to reduce the (pure) exhaustible resource problem to a pure dynamic model that is economicly interesting.

Why infinite horizon?

- If we set a finite planning horizon T, then a rate of consumption of $1/T$ is an obvious plan, and usually a pretty good approximation to the best plan.
- But this naturally uses up all of the stock.
- This is not a problem in inventory management: you just order more for the next planning period. But the definition of *exhaustible* is that once used up, there will never be any more. For any reasonable finite horizon, the question of "what do we do after the resource is used up?" becomes critical—we, and the need that the resource satisfied, will still exist "after."
- Since it's hard to imagine that if you manage to survive to day N, there is *zero* chance of surviving to day $N + 1$, it becomes natural to consider an infinite horizon (this is the *principle of mathematical induction*).

Infinite Horizon Models

- We prefer models with an infinite horizon because they correspond to an autonomous recursive model. That is, we know that tomorrow is mostly like today.
- In particular, just as tomorrow follows today, the day after tomorrow follows tomorrow.
- Also, technically there are “end-point effects”: if the calculation ends at time T, then it will give the highest value to consuming all remaining goods at time T. Such behavior is unrealistic.

Preferences in Infinite Horizon Models

- Since human needs change little from time to time, and must be satisfied at each point in time, we use a (additively) *separable representation* of preferences: $U(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} u_t(c_t)$.
- It seems natural to simplify by assuming symmetry, $u_t(c_t) \equiv u(c_t)$ for all t. But there are two difficulties:
  - In steady state, $\sum \bar{u} = \infty$ for some steady state $\bar{u}$, but maximization is impossible because we can't compare infinities!
  - With exhaustible resources, we eventually use them up, but still

    $$U(1, 0, \ldots) = U(0, 1, 0, \ldots) = U(0, 0, 1, 0, \ldots) = \cdots.$$

    We can't decide when to consume!

Preferences with Infinite Horizon (2)

- Both problems can be solved with *discounting*:

$$U(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \delta_t u(c_t),$$

where $\sum_{t=0}^{\infty} \delta_t < \infty$. Usually we put $\delta_t \equiv \delta^t$ (so that $\delta_0 = 1$).

- Other possibilities include
  - *long run average utility*:

$$U(c_0, c_1, \ldots) = \lim_{m\to\infty} \sup_{n>m} \sum_{t=0}^{n} \frac{u(c_t)}{n},$$

  - and the *overtaking criterion*: $(c_0, c_1, \ldots) \succ (c'_0, c'_1, \ldots)$ if and only if there exists n such that $c_t > c'_t$ for all $t > n$.
  - These emphasize the very long run, and so are used for evaluating government policy. They're not appropriate for consumer or business optimizations.

Constraints in Infinite Horizon Models

- Constraints are easier than preferences.
- A bound on total consumption of the resource over time: $\sum_{t=0}^{\infty} c_t \leq X_0$, or
- a recursive constraint as in Solow's growth model:
  $K_{t+1} \leq K_t - D_t + F(K_t, L_t)$ (Solow used a differential equation)
- or period by period constraint $c_t \leq \bar{c}_t$.

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Pure Exhaustible Resources

- Abbreviate *pure exhaustible resource* to *exhaustible resource.*
- Exhaustible resources are *rival* goods.
- They may be non-excludable (ocean fishing), partially excludable (large oil fields), or completely excludable (small oil fields).
  - Socially optimal usage patterns don't vary; they depend only on the stock.
  - Degree of excludability helps determine market structure, and the equilibrium usage patterns are different.
- The basic ideas of dynamic optimization can be seen with minimum technical difficulty in a model of a monopoly business which owns the whole stock of a resource.

Monopoly

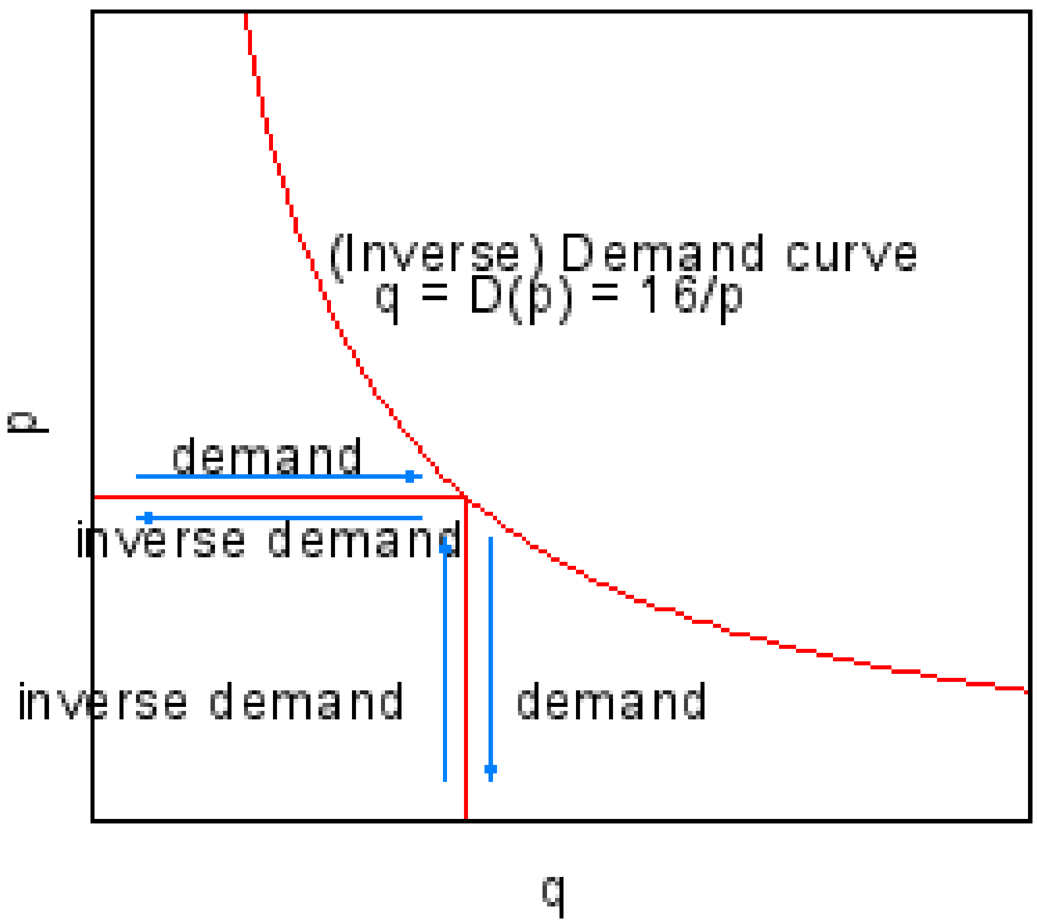
- Single-decision-maker, completely excludable exhaustible resource.
  - Social planner or monopoly; monopoly is simpler (expected discounted value of profit maximization)
  - Unlike intellectual property, consumers are completely excludable.
  - Monopoly is special: other producers are excludable. Government has the power to exclude, too.
- The monopoly decides how much to produce from the *stock* of the resource; the production provides a *flow* of benefits to consumers and is also deducted from the stock. This is similar to the saving decision in growth theory, where produced output (Y) can be used either as consumption ($(1-s)Y$) or savings (sY), and in the latter case it is *added* to the capital stock ($\dot{K}$).
  - We suppose production is costless for simplicity; the usual marginal analysis is possible, of course.
- Critical point: price of the good sold to consumer must equal price of the good saved as asset: they are perfect substitutes.

Consumer Behavior

- Consumer behavior: market demand curve.
  - Consumers do not store the good.
  - The demand curve will be the same for all market structures.
  - Examples: *linear* (constant-slope) and *constant-elasticity* demand curves.
- Assume the market demand function for the resource is constant over time. At each instant of time, the relation between price and total quantity demanded of the resource is the same.
  - After covering the basic theory, we will consider the adjustments that must take place in a growing economy.

Demand

- We denote the instantaneous or one-period *demand curve* by $q = D(p)$. Recall that we also use the *inverse demand curve* $p = D^{-1}(q)$, which has the same graph. The inverse demand curve interpretation is also called the *marginal willingness to pay* curve.



Dynamics and Demand

- With an exhaustible resource, price must eventually rise to choke off demand. Suppose current stock is S_0, and price is never ($i.e$, in no period) greater than $\bar{p}$. Then quantity demanded is never less than $\bar{q} \equiv D(\bar{p})$ since demand is downward sloping. For example, in the figure above, you could take $\bar{p} = 4$, implying $\bar{q} \equiv D(4) = 4$.
- Stock is exhausted no later than time $T \equiv S_0/\bar{q}$. But then at that point price must rise to make quantity demanded equal to 0, to maintain equilibrium.
- The time path price $= \bar{p}$ until stock runs out at time T, then jump to choke price, doesn't make sense. The inverse demand curve shows that the marginal customer at time T has much lower value than the marginal customer at time $T + 1$. Under a plan to exhaust the resource in time T, both a profit-making firm and a social planner want to reduce consumption in time T and increase it (to greater than zero) in time $T + 1$.
  - This applies to any pair of periods. Price rises gradually, forever.

Asset Pricing as Portfolio Choice

- The exhaustible resource is an asset, and is priced by comparing it to other assets, in particular, bonds. We consider whether the firm's future profitability is increased by selling more of the resource and buying more bonds, or by selling less of the resource and buying less bonds (*N.B.* marginal analysis).

- Suppose that at date 0 the market price of the resource is P_0. This is both the price as a commodity (sold to consumers) and as an asset (held for its future value).

- Suppose that the firm can buy or sell bonds with an interest rate of r. (That is, paying 1 yen for a bond today will yield a return of $1 + r$ yen tomorrow.)

Asset Pricing as Portfolio Choice, cont.

- Then the firm faces one of three situations about P_1, the next period price:
  - $P_0 > \frac{1}{1+r} P_1$. The firm should sell more of the resource and invest the cash in bonds. *I.e.*, the resource is overpriced today.
  - $P_0 = \frac{1}{1+r} P_1$. Neither the resource nor bonds increase in value faster; it doesn't matter which the firm holds. *I.e.*, the resource is correctly priced today.
  - $P_0 < \frac{1}{1+r} P_1$. The firm should sell less of the resource and not invest in bonds. *I.e.*, the resource is underpriced today.
- $P_0 > \frac{1}{1+r} P_1$ is derived from comparing two plans

  a. invest P_0 in the resource now, returning P_1 tomorrow;

  b. invest P_0 in bonds now, returning $(1 + r)P_0$ tomorrow;

  the investments are the same, so we just compare the future values:

$$(1 + r)P_0 > P_1.$$

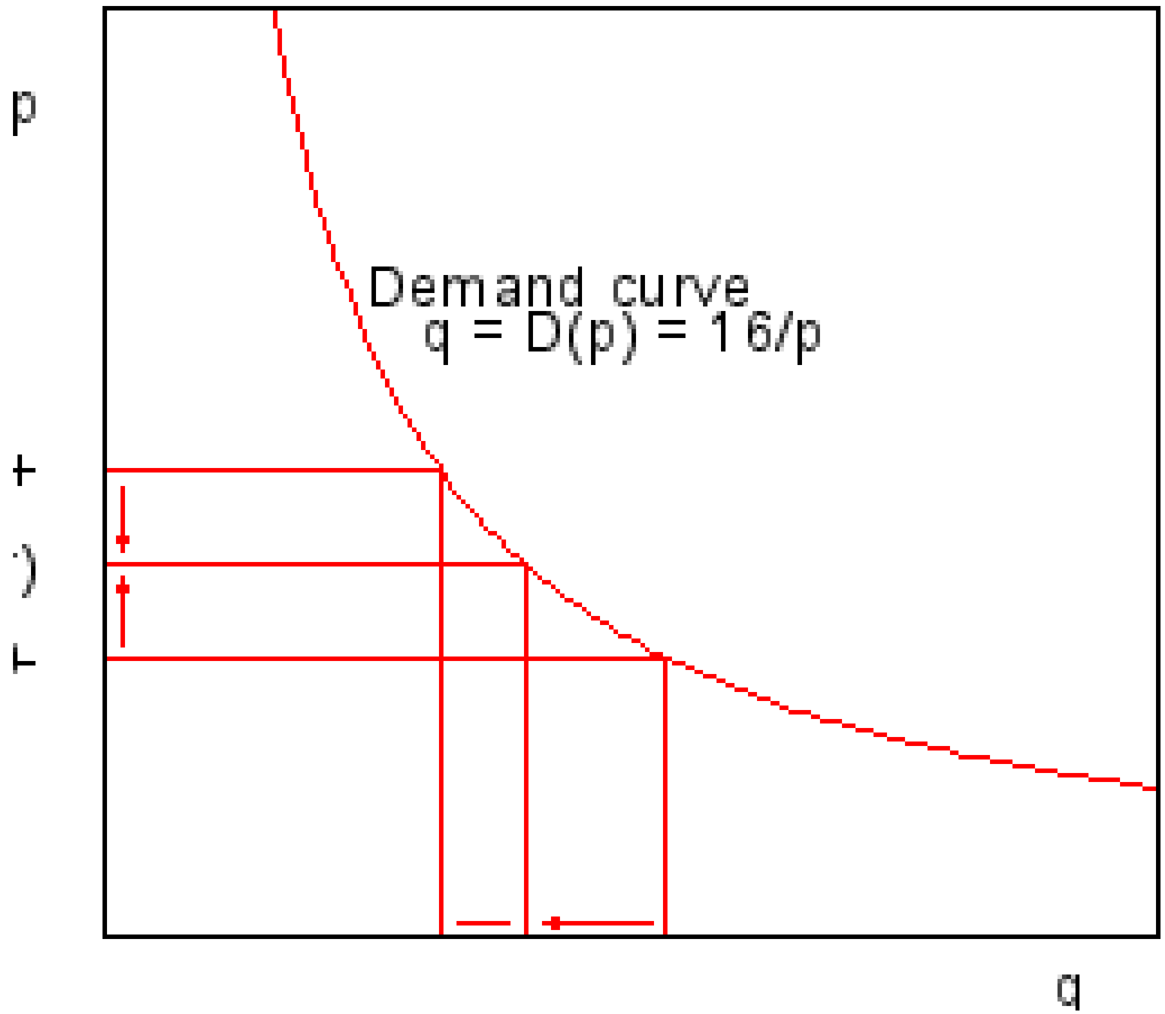
In present value terms ($\frac{1}{1+r}P_1$ is the present value of P_1) the comparison is as above

$$P_0 > \frac{1}{1+r}P_1.$$

- The argument above is called an *arbitrage* argument. *If* there is a sure way to make money by manipulating the financial markets, *then the market is not in equilibrium*. So, we have learned how to evaluate the price path by looking at the *financial market equilibrium*. This is common in dynamic studies.

Interaction of Investment Decision and Market Price

The firm is a monopolist, facing downward sloping demand. Its "investment decision" moves the terms of trade against it.



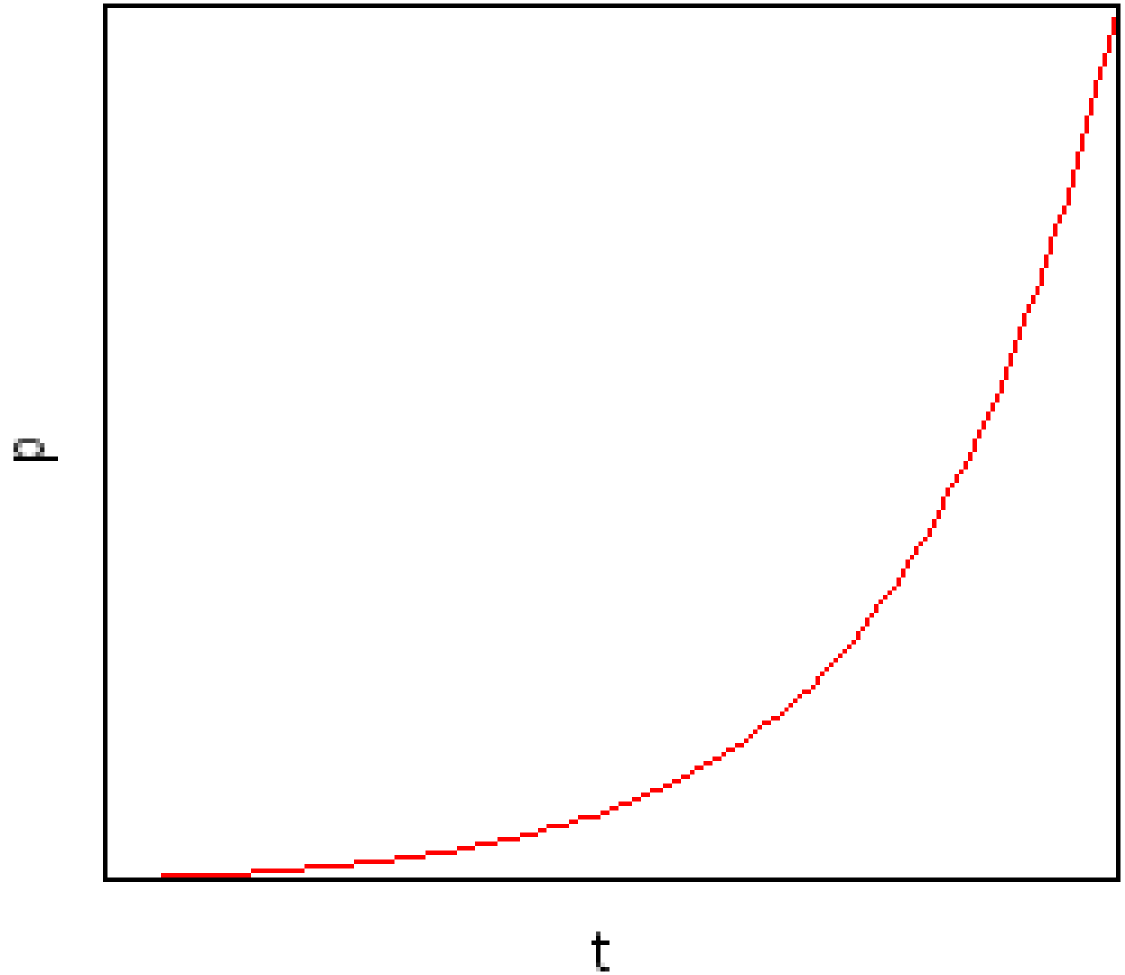
Dynamic Price Path

The firm must expect the price to rise over time according to the rule $P_{t+1} = (1+r)P_t$, *i.e.* $P_t = (1+r)^t P_0$.

Otherwise the firm will want to "play the market," but this also alters the market price of the exhaustible resource, returning the price to this path.

For given P_0, at time t the depletion is $D((1+r)^t P_0)$, and the stock is

$$S_T = S_0 - \sum_{t=0}^{T-1} D((1+r)^t P_0).$$



Initial Price P_0

- For given P_0, at time t the depletion is $D((1+r)^t P_0)$, and the stock is

$$S_T = S_0 - \sum_{t=0}^{T-1} D((1+r)^t P_0).$$

- Obviously price P_0 cannot be too low, or

$$S_0 < \sum_{t=0}^{\infty} D((1+r)^t P_0),$$

  and the stock is used up in finite time. This would drive price to infinity, higher than the path assumed ($(1+r)^t P_0$ is finite).

- What if price P_0 is too high?

$$S_0 - \sum_{t=0}^{\infty} D((1+r)^t P_0) > 0,$$

  and some of the stock is never used. This could happen for inelastic demand.

- Optimal initial price for monopolist depends on elasticity of demand.

Related Results

- If the choke price is not infinite (*e.g.*, with linear demand), in equilibrium the stock will be used up on the date when the price hits the choke price.
  - Interpretation: there is a perfect substitute whose price is the choke price.
  - So when price hits that level switch to the substitute.
- The social planner will surely use up all of the stock at infinity; this results in maximum benefit to society.
- If stocks are excludable, a competitive market in stocks will achieve the social optimum.
- If stocks are not excludable, the tragedy of the commons probably results in overexploitation, and possibly exhaustion in finite time.