Economic Dynamics

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Abstract

Last lecture presented an introduction to Economic Dynamics, the course, the instructor, and the field.
This week we begin discussion of Robert Solow's seminal growth model, after introducing differential equations.

Course Description

This course introduces dynamic analysis in economics. It presumes familiarity with intermediate microeconomics, using optimization theory to explain consumer and firm behavior as a foundation for analysis of economic interactions, primarily via markets.

It is a pure lecture course, with evaluation based on examinations and out-of-class problem sets. Lectures and course materials will be in **English**. Class discussion and questions may be in English or Japanese at your convenience. Out-of-class assignments should be written in English, but may be written in Japanese *with prior approval from the instructor.* Expect substantial delays in evaluation of work written in Japanese without approval.

Administrivia

The following are described in Lecture 1 notes, available from the website:

- Website URL: `http://turnbull.sk.tsukuba.ac.jp/Teach/Dynamics/`
- Evaluation
- Course Resources
  - Primary (required)
  - Background
  - Theory (including mathematical)
  - Applications
- Instructor Information
- Course Information

The Dynamic World

What is the subject of economic dynamics? While many economic problems have a “generic static solution” that can be applied repeatedly in several periods, in many problems linkage must be considered over time. Decisions made now create constraints or opportunities for decisions in future periods.

The importance of dynamics:

- Uncertainty and centrality of time
- Comparative statics is not enough
- Transition dynamics (anticipatory demand) in consumption tax policy
- Complex dynamics

“Classical” Dynamic Topics in Economics

- *Economic growth theory* can help us to understand the context of Japan’s current economic discomfort better – and predict that Korea, Taiwan, and China will experience similar discomfort.
- *Technological innovation* is at the core of modern growth theory; we’ll take a look at dynamics of innovation.
- Do *limitations on natural resources* necessarily imply “limits to growth”? We will look at the economics of exhaustible resources, like oil, and renewable ones, like fish.
- Why are markets so *volatile*, with a persistent but irregular business cycles and financial bubbles that end in “meltdowns”? *Stability analysis* can distinguish inherent instability from external randomness.
- The theory of *pricing of derivative assets* necessarily involves time (as Irving Fisher said).

Basic Ideas of Economic Dynamics

- Dynamics studies an evolving process of change.
- Dynamics often involves irreversibilities.
- Commodities consumed at different *dates* are considered to be different commodities. *E.g.*, “storage” is *production*. Consumers are considered to have *time preference*, or *discounting*.
- Dynamic processes can be very complex, since tomorrow’s outcome of an action today may be outweighed by later outcomes depending on that action. Irreversibility makes it possible to consider simple “endgames”, and analysis works back from there.
- Dynamic optimization may also focuse on *steady states* where the decision-maker’s problem does not change. *I.e.*, the decision in each period recreates the original conditions for the next one.

A Definition of Microeconomics

Microeconomics: *the social science that studies interactions among allocations of scarce resources to competing ends by rational agents.*

Mathematics Used in Economic Dynamics

- The mathematics used in economic dynamic analysis ranges from very basic to horribly advanced. You don't need to worry about the advanced stuff, but it's there if you like math.
- Basic calculus: dynamics is fundamentally concerned with the interplay of *level* and *rate*.
  - In differential calculus, we look at a sequence of levels, or *level as a function of time* ($x(t)$), and derive the *rate of change* as the derivative:

$$\dot{x}(t) = \frac{d}{dt}x(t).$$

  - In integral calculus, we look at a sequence of *rates* ($y(t)$) and derive the level as a function of time:

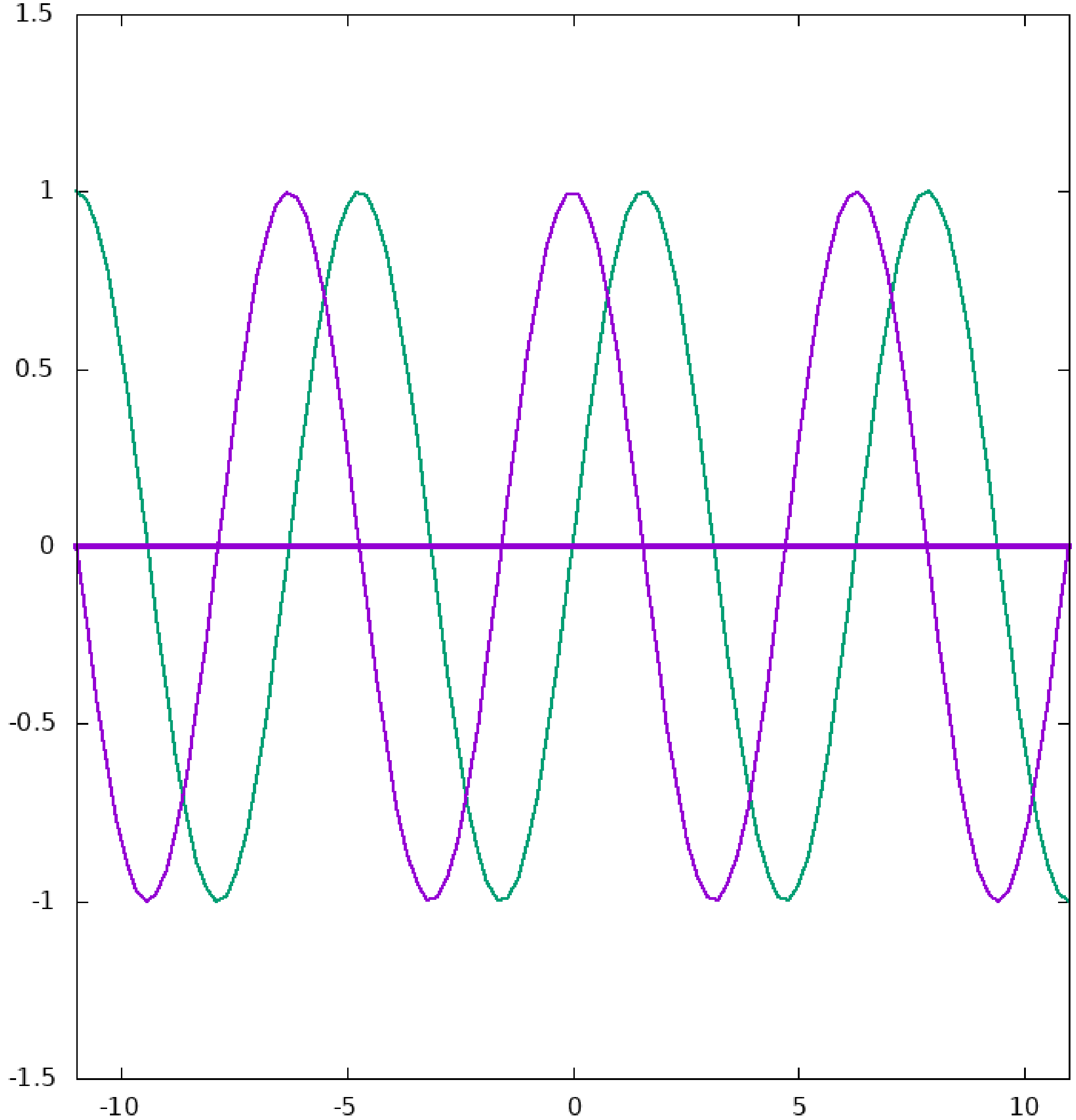
$$Y(t) = \int_{t_0}^{t_1} y(t)dt + Y(t_0).$$

- I'll talk about the harder stuff when we get there.

Differential equations

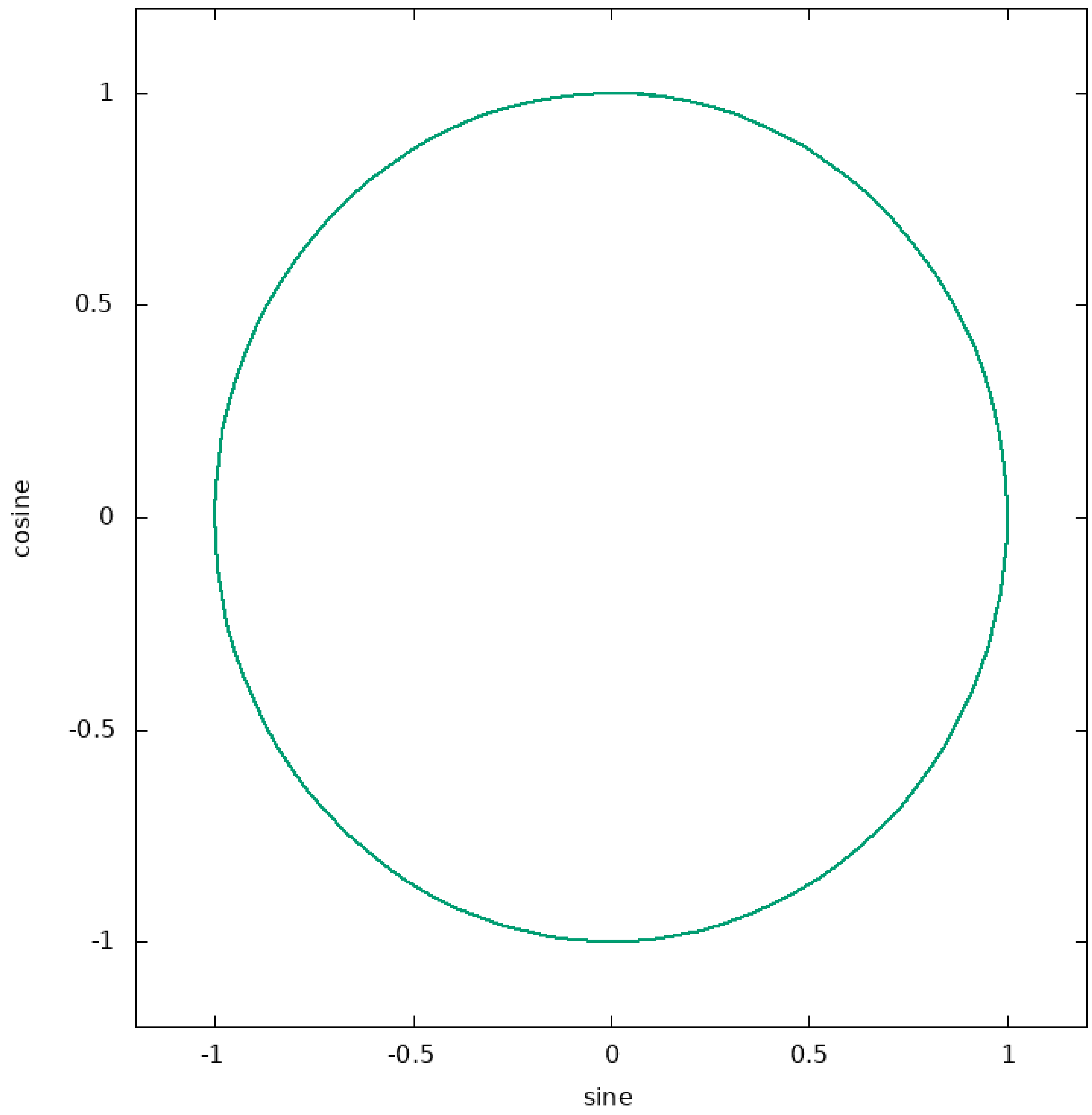
- A *differential equation* is a mathematical expression of a constraint involving certain variable quantities and their derivatives.
  - In dynamics, one variable is the *parameter* and interpreted as *time*.
  - The parameter is often *suppressed* in notation because it's always there: x vs. $x(t)$ (*continuous* time) or x_t (*discrete* time).
- Differential equations can be graphed in many ways: time series, phase diagrams, vector fields, and many others.
- Most useful in in economics is the *phase diagram* of a solution: the combinations of values of variables that occur simultaneously. It does not show *when* combinations, only that they do at some time.
  - Because differential equation solutions are always continuous, the phase diagram displays an *orbit*.
- A vector field shows the direction of "motion" at every point in the phase space, characterizing all solutions in one graph.

Time series: sin x *vs.* cos x



Sine and cosine chase each other (high and low) across the graph.

Phase diagram: sin x *vs.* cos x



Sine and cosine chase each other (high and low) across the graph.

Classifying differential equations

- When the derivative is an ordinary derivative, the equation is called an *ordinary differential equation*.

- Otherwise, partial derivatives are involved, and the equation is called a *partial differential equation*. A system of differential equations containing partial derivatives in any equation is a system of partial differential equations.

- The *order* of a system of differential equations is the order of the highest derivative involved in any equation.
  - Higher-order differential equations may be reduced to first-order differential equations. *E.g.*, $s'' = a$ can be rewritten as $s' = v$ and $v' = a$.

- A differential equation in which the parameter is not mentioned explicitly is called *autonomous*.

Examples

Free fall A *second-order* differential equation $h''(t) = g$.

Soap bubbles The mathematical model of a bubble is based on a system of partial differential equations which characterize equality of air pressure inside and outside of the bubble.

The exponential growth model $\frac{dx}{dt} = ax$.

Logistic growth $\frac{dx}{dt} = x\,(\beta - \delta x)$.

The physical models are *nonautonomous*: the statement of the differential equation requires mentioning the parameter. Both growth models are *autonomous*.

Solving differential equations computationally

- In one sense, a differential equation always has a solution. That is, the fundamental theorem of calculus says that for an integrable function $f(t)$, $\int_\ell^u f(t)dt = F(u) - F(\ell)$, where $f = \frac{dF}{dt}$, and F is continuous and differentiable.

- In practice, we can always compute a time path for $f(t)$ by simulation (picking a value for $F(\ell)$, then setting $F(\ell + (n+1)\delta) = F(\ell + n\delta) + \delta f(\ell + n\delta))$ for δ "sufficiently small").

- However, this is not generally very useful in economic theory (though it is frequently used for examples and actual simulations).

Characterizing solutions

- Because of *resource limitations*, "explosive" growth by individual entities cannot continue indefinitely. From the point of view of individuals in an economy, there should be some stability.
- History shows that individuals *can* usefully predict (near) future conditions by assuming they won't be (much) different from current conditions, so there is a degree of stability.
- For these reasons, not all differential equations are useful models of economic phenomena.
- We also want to be able to characterize the solutions in terms of
  - "where they settle down" (existence of steady states)
  - "how fast they settle down," (stability of and speed of convergence to steady state), and
  - "optimal control" (where the direction and speed of change can be controlled by policy).

Economic Growth Models

- Modern economic growth models focus on
  - increase in output per person
  - that is sustained over time
  - based on **accumulation of capital**.
- What's special about *capital*?
  - Increases in raw materials basically amount to more rapid use of the land; this is not dynamic, since the usage can be decided independently for each instant of time.

    Dynamic in economics means to relate decisions about actions at different times. Saving is dynamic behavior, because it relates sacrifice of consumption now to additional consumption later.

Capital *vs.* other resources

- Of course profitable use of land (or any natural resource) is restricted by available capital or labor, and thus is related to these dynamic processes. But the use of the natural resource itself is linked over time only *indirectly* through other dynamic processes.

- Increase in population is dynamic, but this is determined mostly by biological factors, and we don't really understand economic factors.

- The *capital stock*, which limits the rate at which economic activity occurs, is directly affected by economic factors (*i.e.*, saving and production decisions) and is dynamic because, other things being equal, tomorrow's capital will be the same as today's.

Solow's Contribution

- Robert Solow's contribution was to provide simplifying assumptions about both saving and labor force growth:
  - the labor force grows at a constant rate
  - saving is a constant fraction of income

  and to solve the resulting pure dynamic model.

  "Pure dynamic" means that there is no **micro**economics, in the sense of optimal decisions. The only decision mentioned here is the savings decision, but a fixed savings rate is *a priori* not necessarily optimal.

- Note the focus on the rate of saving. This is very *Keynesian*; many Keynesian models assume that saving is a constant fraction of income. Also, Keynesian models are implicitly intended for government policy-making. The savings rate is affected by government (fiscal and monetary) policy. So the Solow model does provide for government decision-making, but not for private decisions.

Solow's Simplifying Assumptions

- Solow's model (like Marx's) is basically macroeconomic: there is only one produced good, used both for production (in the form of capital) and consumption. A real good that works this way is **rice**: you can eat it, but you can also store it for use as seed in the next planting season.
- Constant labor force growth reflects our poor knowledge about population growth and its basis in economic conditions. We also assume a constant rate of labor force participation (also poorly understood by economists).
- Savings is also poorly understood.

Solow's Dynamic Model

- The central aspect of Solow's model is *capital accumulation*. Denote the rate of capital accumulation by $\dot{K} \equiv \frac{dK}{dt}$.
- $\dot{K} = I - D$, where I and D are investment and depreciation, respectively. Depreciation is assumed to be proportional to the capital stock, K: $D \equiv dK$. *N.B.* d is constant.
- We assume the capital markets are in equilibrium, so that savings equals investment: $I = S$. So far, so trivial.
- In microeconomics, savings depends on many things. Solow (like many Keynesian models) simplified: Saving is a constant proportion of income: $S = sY$.
- Substitution gives the basic accumulation equation: $\dot{K} = sY - dK$, where d and s are constants.

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Compare Marx's Model

Marx's model can be expressed as follows.

- R is capital income, W labor income, and S total saving:

  $$S = s_K R + s_L W = R,$$

  since Marx assumed $s_K = 1$ and $s_L = 0$.

- $R = Y - W$: Profit is what is left over after you pay the workers.
- The subsistence wage $\bar{w}$: a wage just high enough to support a worker and provide for his replacement. Then $W = \bar{w}L$.
- $L = \alpha Y$. Capitalists employ labor proportional to planned production (a fixed-proportions ("Leonteif") production function).
- The rest of the Marxian model is similar to Solow's.

Problems of Marx's Model

- Unfortunately for Marx, both his model of social saving ("capital accumulation") and of labor force growth were incorrect.
- Workers *do* save.
- Workers *do* get richer over time. They do not reproduce until competition causes wages to fall to starvation levels.
- In Marx's model, the fraction of income saved is always increasing, pushing interest down (the "capitalist crisis"). But in reality, the fraction of income going to capital and labor is about constant (Marx could not know that, reliable statistics are only available back to the time when he was writing, and weren't assembled for 50 years or so afterward).

Production in Solow's Model

- The basic accumulation equation was $\dot{K} = sY - dK$.
- K is explained by itself and income. Income (output, Y) is all we need to handle now. But where's population growth?
- Well, of course output will be related to the capital stock through a production function, and the labor force will enter there, too:

  $$Y = F(K, L).$$

  That doesn't look very good; we've now introduced a new variable, and a possibly complex function as well!
- For welfare analysis, income, at least, should be in per capita form.
- If F exhibits *constant returns to scale* (CRTS), all these issues can be simplified.

Constant Returns to Scale

- The mathematical statement of constant returns to scale of F is that F is *linearly homogeneous*, *i.e.*, for any numbers X, Y, and λ,

$$F(\lambda X, \lambda Y) = \lambda F(X, Y).$$

- A constant returns to scale production function can be expressed in a *per-capita form*

$$Y = F(K, L) = F(L \cdot \frac{K}{L}, L \cdot 1) = LF(\frac{K}{L}, 1),$$

where the second equality is trivial, and the third inequality follows by substituting $X \leftarrow \frac{K}{L}$, $Y \leftarrow 1$, and $\lambda \leftarrow L$ in the equation defining linear homogeneity. Now

$$\frac{Y}{L} = F(\frac{K}{L}, 1).$$

Per-Capita Notation

- We simplify notation by using lowercase letters for *per capita* equivalents: y (per capita output), k (capital-labor ratio), and $f(k) \equiv F(k, 1)$ (per capita production function). Then we can write $Y = Lf(k)$ and $y = f(k)$.
- We also need c (per capita consumption) for welfare analysis.
- Note that the *per capita production function* $f(k)$ has only one argument; this can be done since in the per capita equation the second argument of $F(k, 1)$ is a constant.

The Capital-Labor Ratio

- Recall the basic accumulation equation $\dot{K} = sY - dK$.
- Dividing by L gives $\frac{\dot{K}}{L} = s\frac{Y}{L} - d\frac{K}{L} = sy - dk$.
- It is not true that $\dot{k} = \frac{\dot{K}}{L}$; in fact

$$\dot{k} = \frac{\dot{K}}{L} - \frac{\dot{L}}{L}k = \frac{\dot{K}}{L} - nk$$

  where $n \equiv \frac{\dot{L}}{L}$ is the labor force growth rate. (It may help to rewrite the equation using $\frac{d}{dt}X$ instead of $\dot{X}$ notation for each variable.) n is another constant, a *technical* assumption.

- "Technical" means that the "nature" of the solution doesn't change, but (1) the calculations are more difficult and (2) the resulting equation is not straightforward to interpret in terms of economics (you end up saying "ignore X, and think Y ", where X is exactly the complication introduced by variable n, and Y the result from assuming n constant).

The Per-Capita Forms of the Accumulation Equation

- Substituting and rearranging gives the *per capita* form of the basic accumulation equation:

$$\dot{k} = sy - (n + d)k.$$

  – The sy is just the individual worker's investment in his own productivity.

  – The $-nk$ is interesting. If $n > 0$, the labor force is growing. Think of "new workers" entering the labor force: where do they get their capital? It is as if "old" workers share n% of their capital with the "new" ones.

  – From the mathematical point of view, it's no harder to handle $\dot{k} = sy - (n + d)k$ than $\dot{k} = sy - nk$. So it's easy to add depreciation.

- If we substitute $f(k)$ for y, we get the *characteristic equation* of the economy:

$$\dot{k} = sf(k) - (n + d)k,$$

which is an ordinary differential equation.

Conditions on the Production Function

Solow assumed several things about the production function. These are called *technical assumptions* because they are made mostly to make the model tractable and simple. Some are unrealistic, others unnecessary. The technical part is that he assumes that *for all* $K > 0$ *and* $L > 0$,

1. $F(K, 0) = F(0, L) = 0$. Capital and labor are *necessary*.

2. $F_K(K, L) > 0$ and $F_L(K, L) > 0$. Capital and labor are *productive*.

3. $F_{KK}(K, L) < 0$ and $F_{LL}(K, L) < 0$. (More precisely, F is quasi-concave.) Capital and labor are subject to *diminishing marginal returns*.

4. F exhibits *constant returns to scale*.

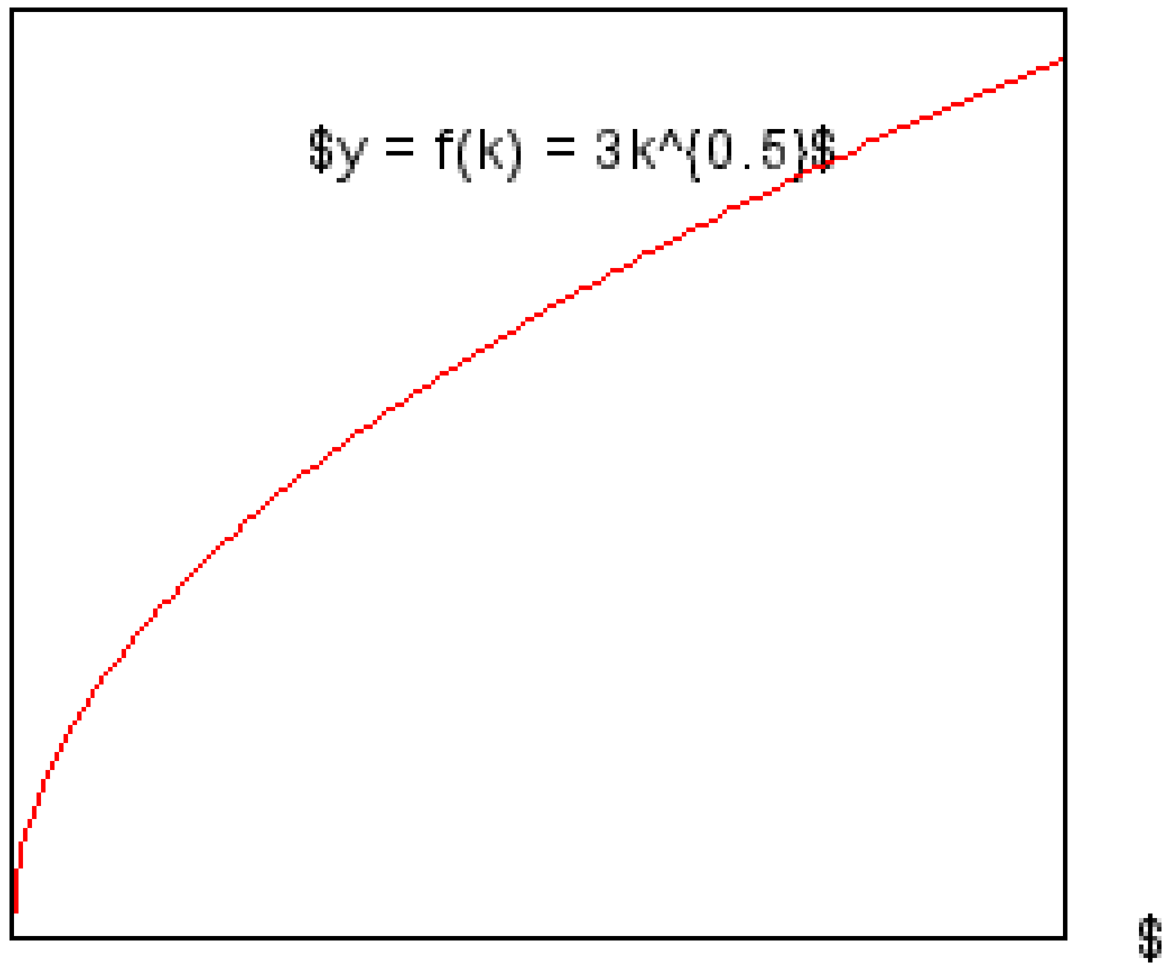
5. $\lim_{k\to 0} F_K(K, 1) = \infty$ and $\lim_{k\to\infty} F_K(K, 1) = 0$, called the *Inada conditions*.

The Production Function, Revisited

- The (per capita) production function must satisfy 3 conditions to make economic sense. Most candidates for production functions do.
  1. $f(0) = 0$.
  2. $f'(k) > 0$, for all $k \geq 0$.
  3. $f''(k) < 0$, for all $k \geq 0$.
- The Inada Conditions are convenient to avoid making careful checks in theoretical arguments:
  4. $lim_{k \to 0} f'(k) = \infty$.
  5. $lim_{k \to \infty} f'(k) = 0$.

The Cobb-Douglas Production Function

A production function satisfying all five conditions is the *Cobb-Douglas* production function $Y = F(K, L) = AK^{\alpha}L^{1-\alpha}$ which has the per capita form $y = f(k) = Ak^{\alpha}$.



The Characteristic Equation

- With a CRTS production function, we can rewrite the basic accumulation equation in per capita form:

$$\dot{k} = sf(k) - (n + d)k,$$

  an autonomous differential equation.

- The solution $k(t)$ to this differential equation determines everything about the economy. $\frac{\dot{L}}{L} = n$ implies $L(t) = L_0 e^{nt}$, and all other variables are given by multiplying by L (*e.g.*, $K(t) \equiv k(t)L(t)$), or from a model equation (*e.g.*, $S(t) = sY(t)$). Thus this equation is the *characteristic equation* of the dynamic system.

- There are *two* variables in this equation for each time t: $k(t)$ and $\dot{k}(t)$. So we also need an initial condition:

$$k(0) = k_0.$$

The State Variable k

- In the characteristic equation $\dot{k} = sf(k) - (n+d)k$ the two variables are closely related: $\dot{k}$ is the time derivative of k.
- So the characteristic equation determines $\dot{k}$ from k, but then $\dot{k}$ determines the "next" k and that implicitly controls the "next" $\dot{k}$, *etc.* So if you know $k(t)$ for any t, you can compute it for *all* t.
- Even more, from k you can compute y using f, from y you can compute $c = (1-s)y$, and the various macro variables from these variables and L, which is exogenous.
- So k is the *state variable* of the system. It is a "sufficient statistic" for all of the information about the system.
  - L is excluded from the state in order to allow the state to be steady.
  - This is theoretically acceptable because L is exogenous, and because F is CRTS, which allows us to "separate" f from L.

Steady States

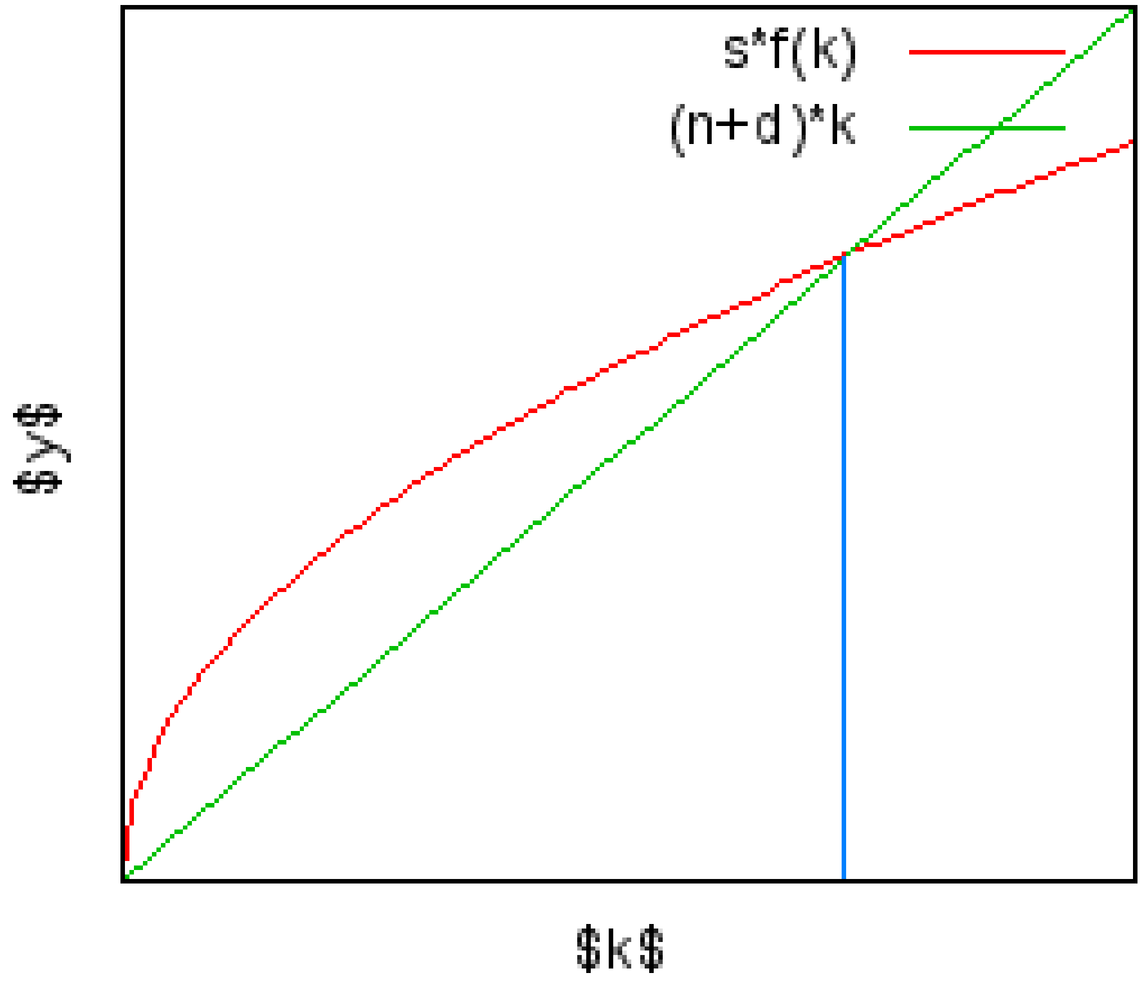
- The most useful benchmark is
$$\dot{k} = 0,$$
not any value of k itself. k does not change, so $y = f(k)$ and $(1 - s)y$ (consumption) do not change either.

  This is a *steady state*.

- The macro economy grows, since all macro variables are multiplied by $L(t)$. A steady state is not "equilibrium" in the usual sense.

- Since everything grows in *proportion* to a single variable, this is *balanced growth*.

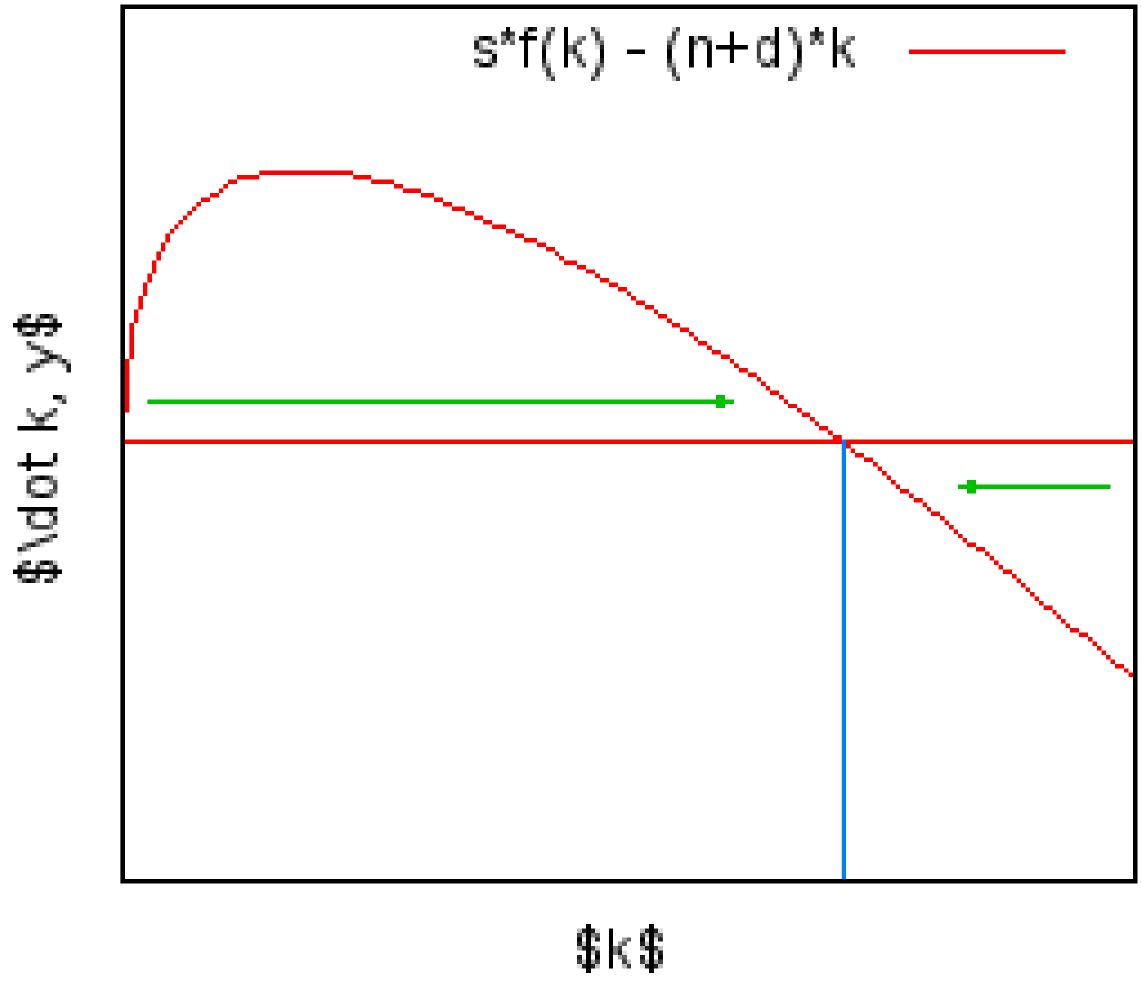
Phase Diagram of Steady State



Characteristics of the Steady State

- Denoted k^*, it occurs at the intersection of the depreciation line and the saving curve, which implies $\dot{k} = 0$.
- It is *stable*. Note that for $k < k^*$, $\dot{k} > 0$, so k rises until it gets to k^*. For $k > k^*$, $\dot{k} < 0$, so k falls until it gets to k^*. ("Stable" means that if the state is forced away from the steady state, it will return to steady state.)
- There is another state where $\dot{k} = 0$: $k = 0$. This is a steady state, but it is not stable.

Stability in the Phase Diagram



Comparative Statics and Policy

- The rate of depreciation is exogenous. Intervention in population is controversial, since raising steady state k^* and y requires *decreasing* population growth.
- The savings rate can be influenced by monetary policy (raising the rate of interest) or by taxing consumption. Though unpopular with today's consumers, if they value the future or their descendants, such policies might be politically feasible.
- Increasing the saving rate always increases steady state k^* and y. But those are implausible policy goals, because y is bounded above as s goes to 1, implying that c (and C) goes to 0.
- The usual, typically individualistic, policy goal analyzed is to maximize per capita consumption in the steady state.

The Golden Rule of Optimal Accumulation

- This goal is achieved by choosing s^* so that

$$f'(k^*) = n + d.$$

- This is an interesting marginal condition: the marginal productivity of capital is exactly used up by counteracting capital thinning.
- Since the maximum exists, there is an absolute cap to per capita consumption. (This is also true of per capita output, but that is less interesting since it involves zero consumption.)
- All of this implies the *Convergence Hypothesis*: in the long run countries will *converge* to similar levels of per capita income, capital stock, and consumption.

Proof of the Golden Rule

1. The goal is to *maximize* c in the steady state, that is, *under the constraint* $0 = sf(k) - (n+d)k$.

2. Although k is a function of s in the steady state, we have the definition $c = (1-s)y$ and the constraint $0 = sf(k) - (n+d)k$. Rewriting the objective as $(1-s)f(k)$ and substituting $sf(k) = (n+d)k$ from the constraint allows us to eliminate explicit mention of s from the objective:

$$f(k) - (n+d)k.$$

3. Differentiating by k gives the first order condition

$$0 = f'(k) - (n+d),$$

   and rearranging proves the result.

Apologia: In More Detail

Note that this procedure is OK because the Inada conditions guarantee the constraint on k can be satisfied for a given s, and if we treat k as a function of s the chain rule gives the first order condition

$$0 = f'(k(s))k'(s) - (n+d)k'(s),$$

and the $k'(s)$ factors out. From the comparative statics we can see that $k'(s) > 0$, so it is the same first order condition.

Lagrangean Proof of the Golden Rule

1. The goal is to *maximize* c in the steady state, that is, *under the constraint* $0 = sf(k) - (n+d)k$.

2. Choosing k and s simultaneously, we write the Lagrangean

$$\mathcal{L}(s,k,\lambda) = (1-s)f(k) + \lambda(sf(k) - (n+d)k).$$

3. The first order conditions are

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial \lambda} = (sf(k) - (n+d)k) \\
0 &= \frac{\partial \mathcal{L}}{\partial s} = -f(k) + \lambda f(k) \\
0 &= \frac{\partial \mathcal{L}}{\partial k} = (1-s)f'(k) + \lambda\left(sf'(k) - (n+d)\right)
\end{aligned}$$

Lagrangean Proof of the Golden Rule

4. The second condition has a trivial solution for λ, substituting that in the third allows elimination of s, and solving the first for s gives the recursive system

$$\begin{aligned} \lambda &= 1 \\ f'(k) &= n + d \\ s &= \frac{(n+d)k}{f(k)} \end{aligned}$$

5. The second equation in point 4 is the Golden Rule.

Homework 3: October 26, 2018

Consider the *linear production function*:

$$F(K, L) = rK + wL.$$

1. Does the linear production function have *constant returns to scale*? Prove your answer.

2. Is the linear production function *neoclassical*? Show your work, and be careful about treatment of equalities!

3. The linear production function is not often used (at least not in this form) in growth theory. Why is this function uninteresting to economists? (Thought question: That is, any guess is a good guess, but please do try to answer this question.)

4. Why do you think I chose the coefficients r and w in the equation defining the linear production function? (Another "thought question.")

Representing Technological Progress

Basically, technology is represented as "enhanced" inputs or outputs, by multiplying them by a factor we may call "productivity." This factor increases over time.

- To derive a tractable *characteristic equation* for technological progress, it is easiest to assume *labor-enhancing* progress, with a production function of the form $F(K, A(t)L)$. We also call this *Harrod-neutral* progress.
  - Define *effective labor* by $\tilde{L} \equiv AL$.
  - Define $\tilde{y} \equiv \frac{Y}{AL}$ and $\tilde{k} \equiv \frac{K}{AL}$.
  - Assume A grows at rate $g > 0$ (*i.e.*, $A(t) = A_0 e^{gt}$). Then $\tilde{L} = A_0 e^{gt} L_0 e^{nt} = A_0 L_0 e^{(n+g)t}$ grows at rate $n + g$.
- The steady state is defined by $\dot{\tilde{k}} = 0$, where the characteristic equation is

$$\dot{\tilde{k}} = sf(\tilde{k}) - (n + g + d)\tilde{k}.$$

Balanced Growth with Technological Progress

- Since the saving function hasn't changed and the capital thinning function is still linear, with the Inada conditions the analysis is the same as in the basic Solow model: there is a stable steady state $\tilde{k}^* > 0$.
- The comparative statics of $\tilde{k}^*$ is the same as before with respect to all variables, and g has the same properties as n or d.
- But note that $k = A\tilde{k}$ and $y = A\tilde{y}$, so $y^* = \frac{Y}{L}$ and $k^* = \frac{K}{L}$ grow at rate $\frac{\dot{A}}{A} = g > 0$ in the steady state where $\tilde{k}^*$ does not grow.
- Notice that by redefining the state variable from k to $\tilde{k}$ (and assuming the "right" kind of technological progress), we can easily extend the analysis in this way.

Other Representations of Technology

- Growth accounting (presented next) is based on *Hicks-neutral* technological progress, using a production function of the form $A(t)F(K, L)$ where A is the technology index and F is a constant returns to scale production function.

- For completeness, we can define *Solow-neutrality*, of the form $F(A(t)K, L)$. This is a plausible representation of the kind of technological progress produced by "R&D."

- Of course, we can combine all three in theory. That might be the most accurate representation of reality, but in practice each kind of progress is useful in different applications.

Endogenous Technological Progress

- We won't be able to give a full growth model with endogenous technological progress; it's too complex.
- First, note that the macro models are missing a very important idea: physical capital becomes obsolete as well as breaking down.
- We could model this as a higher rate of depreciation d.

  But: d depends on $\dot{A}$ (the change in "technology level").
- Alternatively, we could use a "vintage" model, in which only the capital bought at time t has productivity $A(t)$.

  But: we either must associate labor with specific vintages of capital (complicated!), or we must use a model with Solow-neutral technical progress—which neither provides a base for simple growth accounting, nor is easily adapted to the Solow method for analyzing growth models.

Multi-Sector Models

- Simplest way to make d endogenous is a two-sector model, with one sector "producing" technology (R&D).
- This is dynamic: current investment constrains the future "stock" of technology (it must grow).
- It is irreversible.
- Compared to Solow model, basic result (a steady-state balanced growth path exists) is the same.
  - The Japanese economist *Hirofumi Uzawa* is most famous for this analysis.
- The math is more complicated. Stability of steady-state balanced growth path depends on the parameter comparing the production functions for physical capital and technology.

Homework 4: October 26, 2018

This exercise concerns *dated commodities* and *dated technology (production functions).*

1. Let $F_t(K, L)$ denote Toyota Company's production function for cars in year t.
   (a) What does $F_{2006}(K, L) < F_{1996}(K, L)$ mean?
   (b) Do you think the inequality in (a) is a historical fact? Explain why or why not.
   (c) What does $F_{2006}(K_{2006}, L_{2006}) > F_{1996}(K_{1996}, L_{1996})$ mean?
   (d) Ignoring historical facts about Toyota, is it logically possible for both (a) and (c) to be true? Explain why or why not.

Homework 5: October 26, 2018

1. Consider the Cobb-Douglas production function with *Hicks-neutral* technological progress:

$$F(K, L, \lambda) = A_0 e^{\lambda t} K^{\alpha} L^{1-\alpha}.$$

   (a) Show that for appropriate constants B_0 and β the Hicks-neutral technological progress can be expressed as a production function with *Solow-neutral* technological progress:

$$F(K, L, \beta) = B_0 (e^{\beta t} K)^{\alpha} L^{1-\alpha}.$$

   (b) Show that for the appropriate constants C_0 and γ the Hicks-neutral technological progress can be expressed as a production function with *Harrod-neutral* technological progress:

$$F(K, L, \gamma) = C_0 K^{\alpha} (e^{\gamma t} L)^{1-\alpha}.$$

2. Explain why Solow-neutral technological progress is also called *capital-enhancing* technological progress. Use the equation in your answer.

3. Explain why Harrod-neutral technological progress is also called *labor-enhancing* technological progress. Use the equation in your answer.

4. Explain why the method used to adapt Solow's analysis to Harrod-neutral technological progress will *not* work for Solow-neutral technological progress. *Hint: the answer I have in mind uses the idea of* exogenous *and* endogenous *variables. There are probably other good ways to explain it, however.*