Economic Dynamics

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Lecture 3: October 16, 2020

Abstract

Last lecture presented an introduction to Economic Dynamics, the course, the instructor, and the field.

This week we begin discussion of Robert Solow's seminal growth model, after introducing differential equations.

Review of the ideas of economic dynamics

First we review some of the basic ideas of economic dynamics.

- The role of time in economics
- Comparative statics in intermediate microeconomics *vs.* “true” dynamics
- Examples of topics in dynamics economics

Economic Growth Models

- Modern economic growth models focus on
  - increase in output per person
  - that is sustained over time
  - based on **accumulation of capital**.
- What's special about *capital*?
  - Increases in raw materials basically amount to more rapid use of the land; this is not dynamic, since the usage can be decided independently for each instant of time.

    Dynamic in economics means to relate decisions about actions at different times. Saving is dynamic behavior, because it relates sacrifice of consumption now to additional consumption later.

Capital *vs.* other resources

- Of course profitable use of land (or any natural resource) is restricted by available capital or labor, and thus is related to these dynamic processes. But the use of the natural resource itself is linked over time only *indirectly* through other dynamic processes.
- Increase in population is dynamic, but this is determined mostly by biological factors, and we don't really understand economic factors.
- The *capital stock*, which limits the rate at which economic activity occurs, is directly affected by economic factors (*i.e.*, saving and production decisions) and is dynamic because, other things being equal, tomorrow's capital will be the same as today's.

Problems of Marx's Model

- Unfortunately for Marx, both his model of social saving ("capital accumulation") and of labor force growth were incorrect.
- Workers *do* save.
- Workers *do* get richer over time. They do not reproduce until competition causes wages to fall to starvation levels.
- In Marx's model, the fraction of income saved is always increasing, pushing interest down (the "capitalist crisis"). But in reality, the fraction of income going to capital and labor is about constant (Marx could not know that, reliable statistics are only available back to the time when he was writing, and weren't assembled for 50 years or so afterward).

Solow's Contribution

- Robert Solow's contribution was to provide simplifying assumptions about both saving and labor force growth:
  - the labor force grows at a constant rate
  - saving is a constant fraction of income

  and to solve the resulting pure dynamic model.

  "Pure dynamic" means that there is no **micro**economics, in the sense of optimal decisions. The only decision mentioned here is the savings decision, but a fixed savings rate is *a priori* not necessarily optimal.

- Note the focus on the rate of saving. This is very *Keynesian*; many Keynesian models assume that saving is a constant fraction of income. Also, Keynesian models are implicitly intended for government policy-making. The savings rate is affected by government (fiscal and monetary) policy. So the Solow model does provide for government decision-making, but not for private decisions.

Solow's Simplifying Assumptions

- Solow's model (like Marx's) is basically macroeconomic: there is only one produced good, used both for production (in the form of capital) and consumption. A real good that works this way is **rice**: you can eat it, but you can also store it for use as seed in the next planting season.
- Constant labor force growth reflects our poor knowledge about population growth and its basis in economic conditions. We also assume a constant rate of labor force participation (also poorly understood by economists).
- Savings is also poorly understood.

Solow's Dynamic Model

- The central aspect of Solow's model is *capital accumulation*. Denote the rate of capital accumulation by $\dot{K} \equiv \frac{dK}{dt}$.
- $\dot{K} = I - D$, where I and D are investment and depreciation, respectively. Depreciation is assumed to be proportional to the capital stock, K: $D \equiv dK$. *N.B.* d is constant.
- We assume the capital markets are in equilibrium, so that savings equals investment: $I = S$. So far, so trivial.
- In microeconomics, savings depends on many things. Solow (like many Keynesian models) simplified: Saving is a constant proportion of income: $S = sY$.
- Substitution gives the basic accumulation equation: $\dot{K} = sY - dK$, where d and s are constants.

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Production in Solow's Model

- The basic accumulation equation was $\dot{K} = sY - dK$.
- K is explained by itself and income. Income (output, Y) is all we need to handle now. But where's population growth?
- Well, of course output will be related to the capital stock through a production function, and the labor force will enter there, too:

  $$Y = F(K, L).$$

  That doesn't look very good; we've now introduced a new variable, and a possibly complex function as well!
- For welfare analysis, income, at least, should be in per capita form.
- If F exhibits *constant returns to scale* (CRTS), all these issues can be simplified.

Constant Returns to Scale

- The mathematical statement of constant returns to scale of F is that F is *linearly homogeneous*, *i.e.*, for any numbers X, Y, and λ,

$$F(\lambda X, \lambda Y) = \lambda F(X, Y).$$

- A constant returns to scale production function can be expressed in a *per-capita form*

$$Y = F(K, L) = F(L \cdot \frac{K}{L}, L \cdot 1) = LF(\frac{K}{L}, 1),$$

where the second equality is trivial, and the third inequality follows by substituting $X \leftarrow \frac{K}{L}$, $Y \leftarrow 1$, and $\lambda \leftarrow L$ in the equation defining linear homogeneity. Now

$$\frac{Y}{L} = F(\frac{K}{L}, 1).$$

Per-Capita Notation

- We simplify notation by using lowercase letters for *per capita* equivalents: y (per capita output), k (capital-labor ratio), and $f(k) \equiv F(k, 1)$ (per capita production function). Then we can write $Y = Lf(k)$ and $y = f(k)$.
- We also need c (per capita consumption) for welfare analysis.
- Note that the *per capita production function* $f(k)$ has only one argument; this can be done since in the per capita equation the second argument of $F(k, 1)$ is a constant.

The Capital-Labor Ratio

- Recall the basic accumulation equation $\dot{K} = sY - dK$.
- Dividing by L gives $\frac{\dot{K}}{L} = s\frac{Y}{L} - d\frac{K}{L} = sy - dk$.
- It is not true that $\dot{k} = \frac{\dot{K}}{L}$; in fact

$$\dot{k} = \frac{\dot{K}}{L} - \frac{\dot{L}}{L}k = \frac{\dot{K}}{L} - nk$$

  where $n \equiv \frac{\dot{L}}{L}$ is the labor force growth rate. (It may help to rewrite the equation using $\frac{d}{dt}X$ instead of $\dot{X}$ notation for each variable.) n is another constant, a *technical* assumption.

- "Technical" means that the "nature" of the solution doesn't change, but (1) the calculations are more difficult and (2) the resulting equation is not straightforward to interpret in terms of economics (you end up saying "ignore X, and think Y ", where X is exactly the complication introduced by variable n, and Y the result from assuming n constant).

The Per-Capita Forms of the Accumulation Equation

- Substituting and rearranging gives the *per capita* form of the basic accumulation equation:

$$\dot{k} = sy - (n + d)k.$$

  - The sy is just the individual worker's investment in his own productivity.
  - The $-nk$ is interesting. If $n > 0$, the labor force is growing. Think of "new workers" entering the labor force: where do they get their capital? It is as if "old" workers share $n\%$ of their capital with the "new" ones.
  - From the mathematical point of view, it's no harder to handle $\dot{k} = sy - (n + d)k$ than $\dot{k} = sy - nk$. So it's easy to add depreciation.

- If we substitute $f(k)$ for y, we get the *characteristic equation* of the economy:

$$\dot{k} = sf(k) - (n + d)k,$$

  which is an ordinary differential equation.

Conditions on the Production Function

Solow assumed several things about the production function. These are called *technical assumptions* because they are made mostly to make the model tractable and simple. Some are unrealistic, others unnecessary. The technical part is that he assumes that *for all* $K > 0$ *and* $L > 0$,

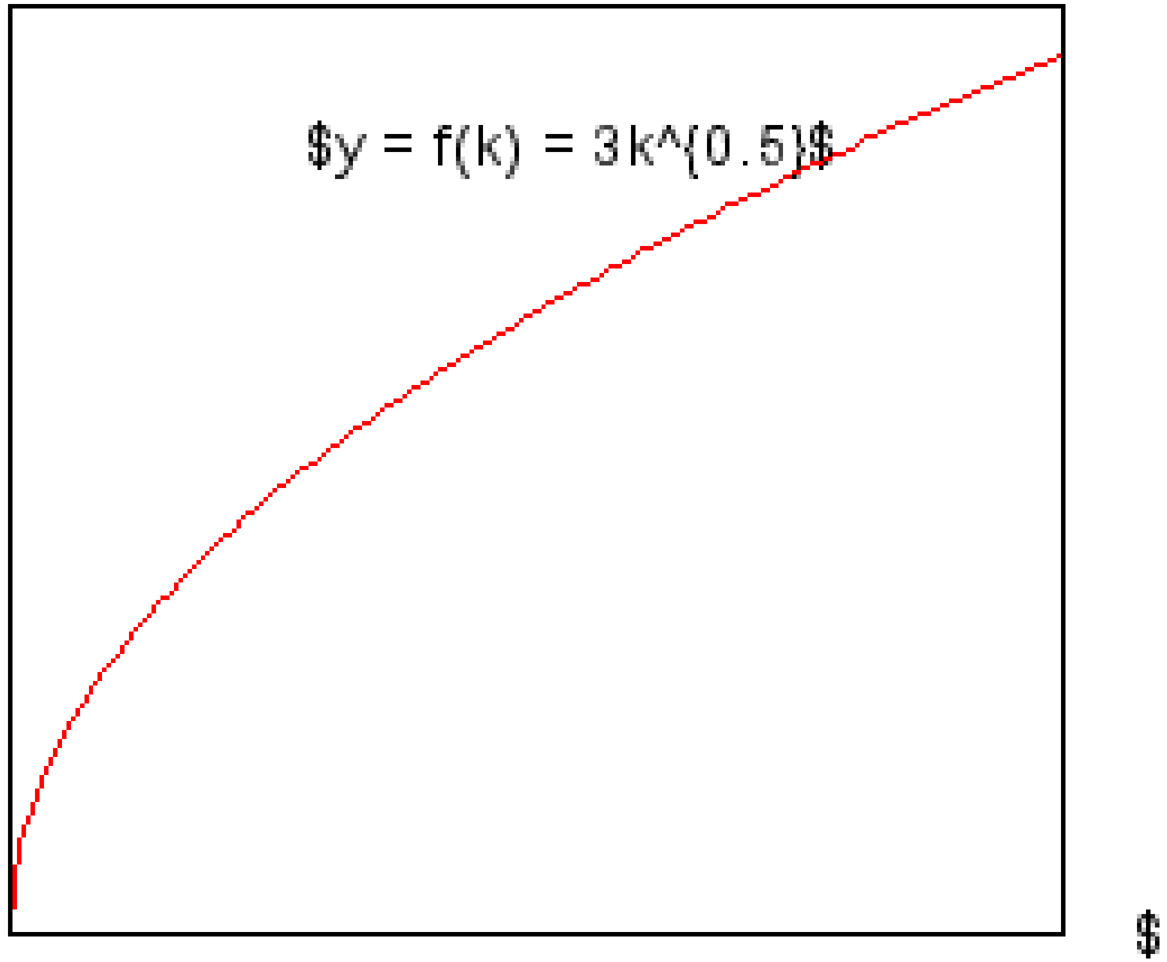
1. $F(K, 0) = F(0, L) = 0$. Capital and labor are *necessary*.
2. $F_K(K, L) > 0$ and $F_L(K, L) > 0$. Capital and labor are *productive*.
3. $F_{KK}(K, L) < 0$ and $F_{LL}(K, L) < 0$. (More precisely, F is quasi-concave.) Capital and labor are subject to *diminishing marginal returns*.
4. F exhibits *constant returns to scale*.
5. $\lim_{k \to 0} F_K(K, 1) = \infty$ and $\lim_{k \to \infty} F_K(K, 1) = 0$, called the *Inada conditions*.

The Production Function, Revisited

- The (per capita) production function must satisfy 3 conditions to make economic sense. Most candidates for production functions do.
  1. $f(0) = 0$.
  2. $f'(k) > 0$, for all $k \geq 0$.
  3. $f''(k) < 0$, for all $k \geq 0$.
- The Inada Conditions are convenient to avoid making careful checks in theoretical arguments:
  4. $lim_{k \to 0} f'(k) = \infty$.
  5. $lim_{k \to \infty} f'(k) = 0$.

The Cobb-Douglas Production Function

A production function satisfying all five conditions is the *Cobb-Douglas* production function $Y = F(K, L) = AK^{\alpha}L^{1-\alpha}$ which has the per capita form $y = f(k) = Ak^{\alpha}$.



The Characteristic Equation

- With a CRTS production function, we can rewrite the basic accumulation equation in per capita form:

$$\dot{k} = sf(k) - (n + d)k,$$

an autonomous differential equation.

- The solution $k(t)$ to this differential equation determines everything about the economy. $\frac{\dot{L}}{L} = n$ implies $L(t) = L_0 e^{nt}$, and all other variables are given by multiplying by L (*e.g.*, $K(t) \equiv k(t)L(t)$), or from a model equation (*e.g.*, $S(t) = sY(t)$). Thus this equation is the *characteristic equation* of the dynamic system.

- There are *two* variables in this equation for each time t: $k(t)$ and $\dot{k}(t)$. So we also need an initial condition:

$$k(0) = k_0.$$

The State Variable k

- In the characteristic equation $\dot{k} = sf(k) - (n+d)k$ the two variables are closely related: $\dot{k}$ is the time derivative of k.
- So the characteristic equation determines $\dot{k}$ from k, but then $\dot{k}$ determines the "next" k and that implicitly controls the "next" $\dot{k}$, *etc.* So if you know $k(t)$ for any t, you can compute it for *all* t.
- Even more, from k you can compute y using f, from y you can compute $c = (1-s)y$, and the various macro variables from these variables and L, which is exogenous.
- So k is the *state variable* of the system. It is a "sufficient statistic" for all of the information about the system.
  - L is excluded from the state in order to allow the state to be steady.
  - This is theoretically acceptable because L is exogenous, and because F is CRTS, which allows us to "separate" f from L.

Steady States

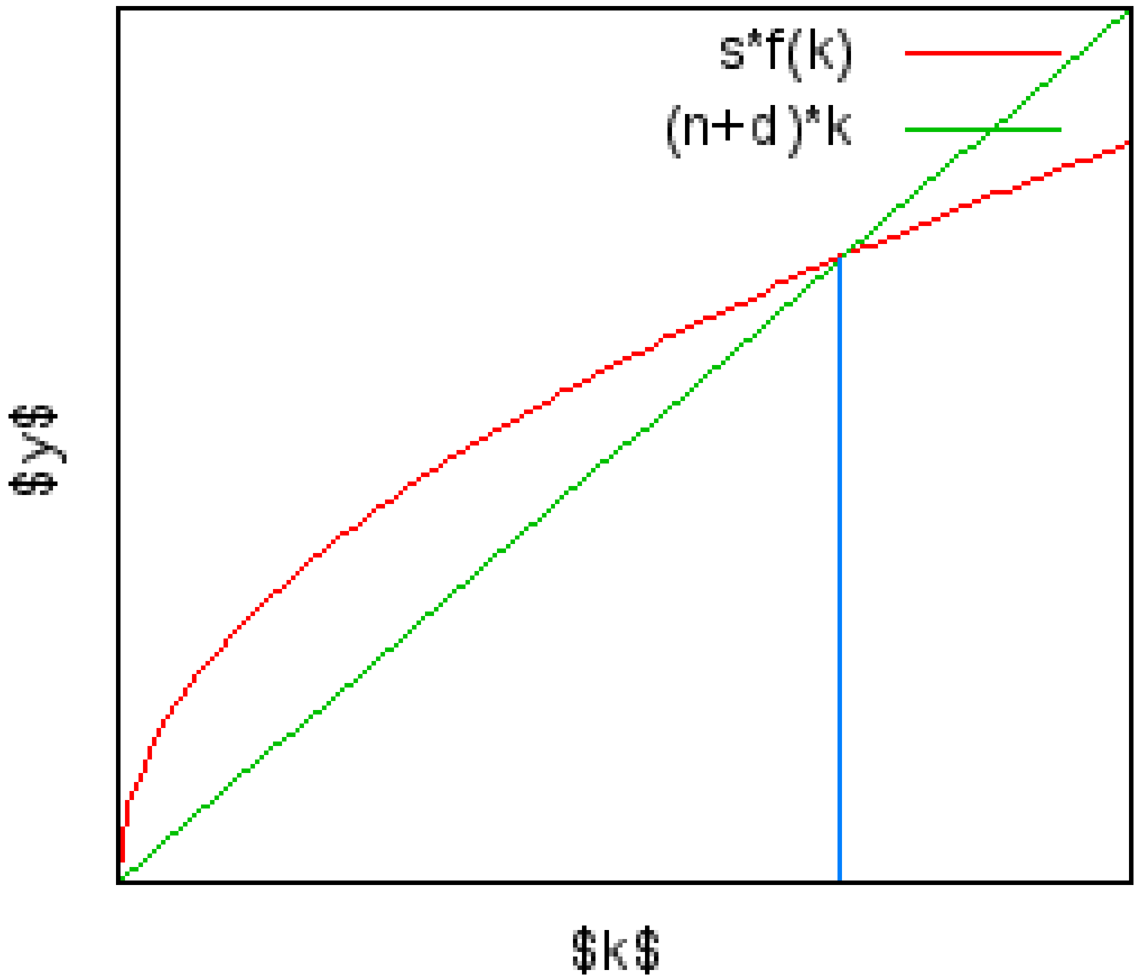
- The most useful benchmark is
$$\dot{k} = 0,$$
not any value of k itself. k does not change, so $y = f(k)$ and $(1 - s)y$ (consumption) do not change either.

  This is a *steady state*.

- The macro economy grows, since all macro variables are multiplied by $L(t)$. A steady state is not "equilibrium" in the usual sense.

- Since everything grows in *proportion* to a single variable, this is *balanced growth*.

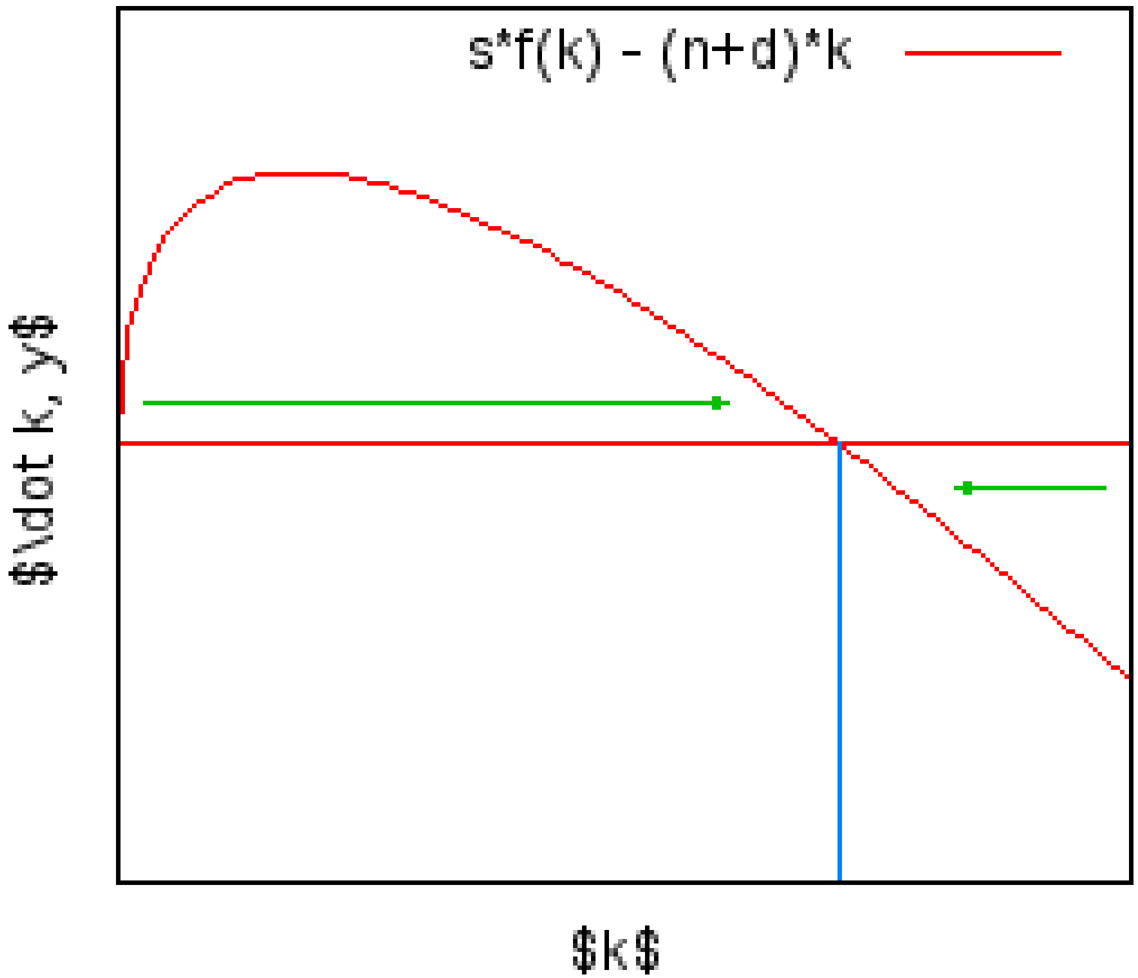
Phase Diagram of Steady State



Characteristics of the Steady State

- Denoted k^*, it occurs at the intersection of the depreciation line and the saving curve, which implies $\dot{k} = 0$.
- It is *stable*. Note that for $k < k^*$, $\dot{k} > 0$, so k rises until it gets to k^*. For $k > k^*$, $\dot{k} < 0$, so k falls until it gets to k^*. ("Stable" means that if the state is forced away from the steady state, it will return to steady state.)
- There is another state where $\dot{k} = 0$: $k = 0$. This is a steady state, but it is not stable.

Stability in the Phase Diagram



Comparative Statics and Policy

- The rate of depreciation is exogenous. Intervention in population is controversial, since raising steady state k^* and y requires *decreasing* population growth.
- The savings rate can be influenced by monetary policy (raising the rate of interest) or by taxing consumption. Though unpopular with today's consumers, if they value the future or their descendants, such policies might be politically feasible.
- Increasing the saving rate always increases steady state k^* and y. But those are implausible policy goals, because y is bounded above as s goes to 1, implying that c (and C) goes to 0.
- The usual, typically individualistic, policy goal analyzed is to maximize per capita consumption in the steady state.

The Golden Rule of Optimal Accumulation

- This goal is achieved by choosing s^* so that

$$f'(k^*) = n + d.$$

- This is an interesting marginal condition: the marginal productivity of capital is exactly used up by counteracting capital thinning.
- Since the maximum exists, there is an absolute cap to per capita consumption. (This is also true of per capita output, but that is less interesting since it involves zero consumption.)
- All of this implies the *Convergence Hypothesis*: in the long run countries will *converge* to similar levels of per capita income, capital stock, and consumption.

Proof of the Golden Rule

1. The goal is to *maximize* c in the steady state, that is, *under the constraint* $0 = sf(k) - (n + d)k$.

2. Although k is a function of s in the steady state, we have the definition $c = (1 - s)y$ and the constraint $0 = sf(k) - (n + d)k$. Rewriting the objective as $(1 - s)f(k)$ and substituting $sf(k) = (n + d)k$ from the constraint allows us to eliminate explicit mention of s from the objective:

$$f(k) - (n + d)k.$$

3. Differentiating by k gives the first order condition

$$0 = f'(k) - (n + d),$$

and rearranging proves the result.

Apologia: In More Detail

Note that this procedure is OK because the Inada conditions guarantee the constraint on k can be satisfied for a given s, and if we treat k as a function of s the chain rule gives the first order condition

$$0 = f'(k(s))k'(s) - (n + d)k'(s),$$

and the $k'(s)$ factors out. From the comparative statics we can see that $k'(s) > 0$, so it is the same first order condition.

Lagrangean Proof of the Golden Rule

1. The goal is to *maximize* c in the steady state, that is, *under the constraint* $0 = sf(k) - (n+d)k$.

2. Choosing k and s simultaneously, we write the Lagrangean

$$\mathcal{L}(s,k,\lambda) = (1-s)f(k) + \lambda(sf(k) - (n+d)k).$$

3. The first order conditions are

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial \lambda} = (sf(k) - (n+d)k) \\ 0 &= \frac{\partial \mathcal{L}}{\partial s} = -f(k) + \lambda f(k) \\ 0 &= \frac{\partial \mathcal{L}}{\partial k} = (1-s)f'(k) + \lambda\,(sf'(k) - (n+d)) \end{aligned}$$

Lagrangean Proof of the Golden Rule

4. The second condition has a trivial solution for λ, substituting that in the third allows elimination of s, and solving the first for s gives the recursive system

$$\begin{aligned} \lambda &= 1 \\ f'(k) &= n + d \\ s &= \frac{(n+d)k}{f(k)} \end{aligned}$$

5. The second equation in point 4 is the Golden Rule.

Rich and Poor

- The central question of *development economics* is "Why Are (We) So Rich and (They) So Poor?"
  – Asking the question implies differences in economic status.
  – Why are there differences in economic status?
- Possible explanations:
  – Differing factor endowments (educated labor, natural resources).
  – Different social organization (*e.g.*, the market).
  – Different technology (engineers and installed plant).
- But no one factor explains everything well.

The Convergence Hypothesis

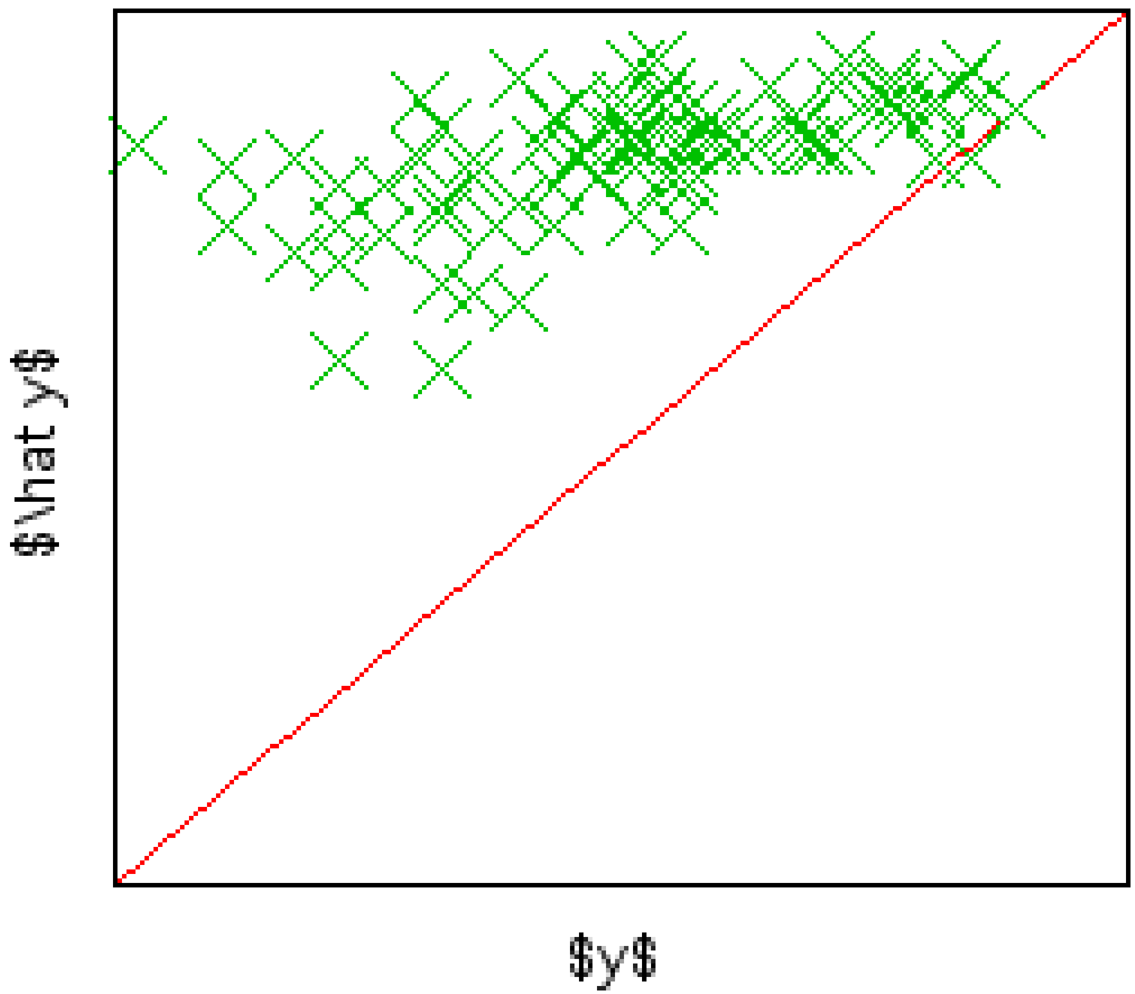
- The *convergence hypothesis* states that in the long run countries will *converge* to similar levels of per capita income, capital stock, and consumption.
  - Macro convergence will not occur, because populations differ.
- The convergence hypothesis depends on common values of f, d, n, and s.
  - f and d are transferable technological parameters.
  - The *demographic transition* is common experience: n should converge.
  - s depends on culture. However, policies to maximize per-capita consumption in the long run will induce the same value of s^* according to the Golden rule.
- Empirical justification for the convergence hypothesis:
  - Technology transfer is occurring.
  - Diminishing returns to capital implies poor countries have higher MP_K, attracting investment.
  - Low income countries are also far below k^*, implying high $\dot{k}$.

Transition Dynamics

- With poor medical care, birth rates and death rates are high, population growth rate low.
- With improvements in health, death rates drop quickly, resulting in high population growth, but this is followed by the demographic transition to low birth rates.
  - Advanced countries generally have population growth determined mostly by immigration rate.
- As economic welfare converges (the convergence hypothesis) the primary motivation for migration is removed, so convergence is self-stabilizing.
- Poor countries have higher growth rates
- Poor countries catch up in absolute terms

Cross-Section Comparison of Predicted Income

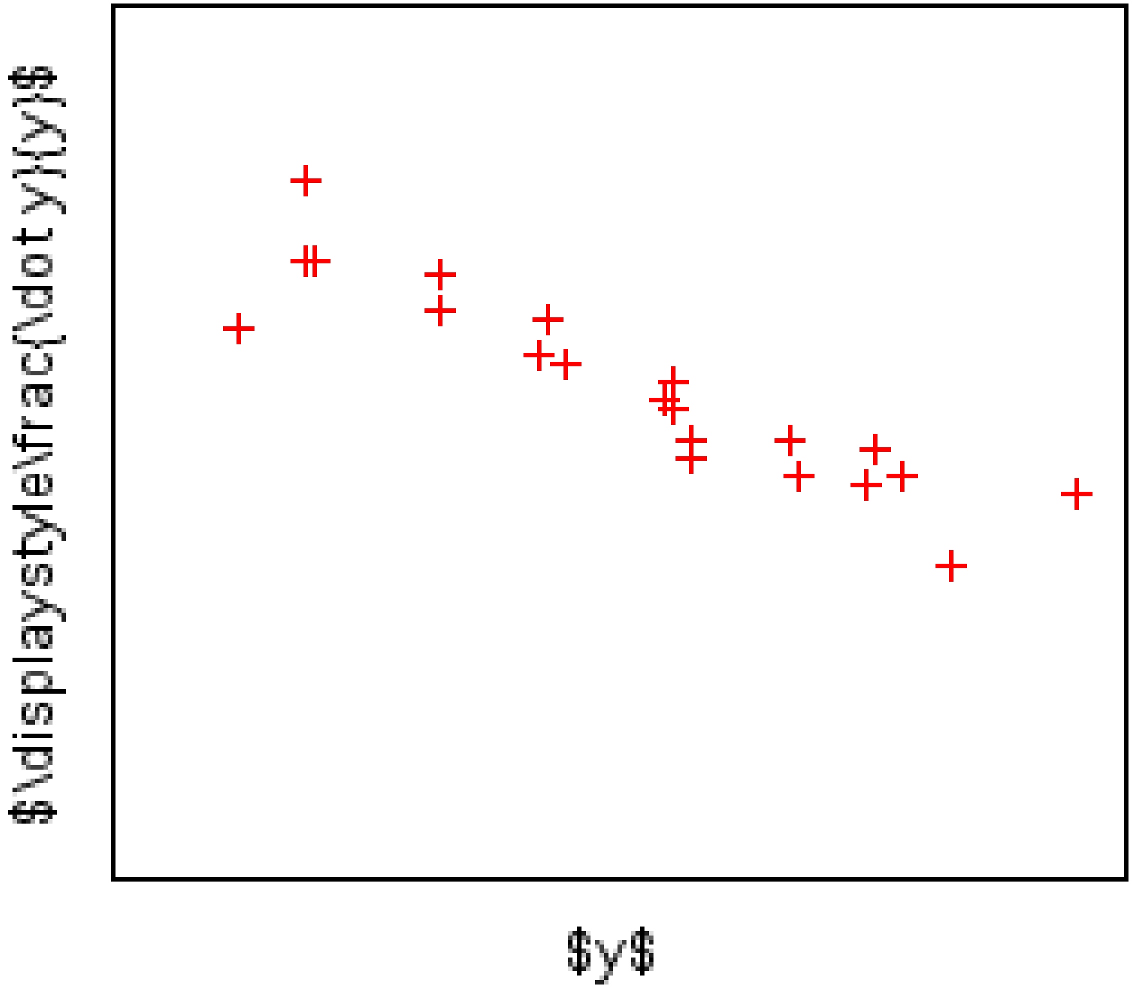
Estimate the production function $f(k)$, predict $\hat{y}$, and compare to y. Predictions are way too high for poor countries (inefficient use of capital).



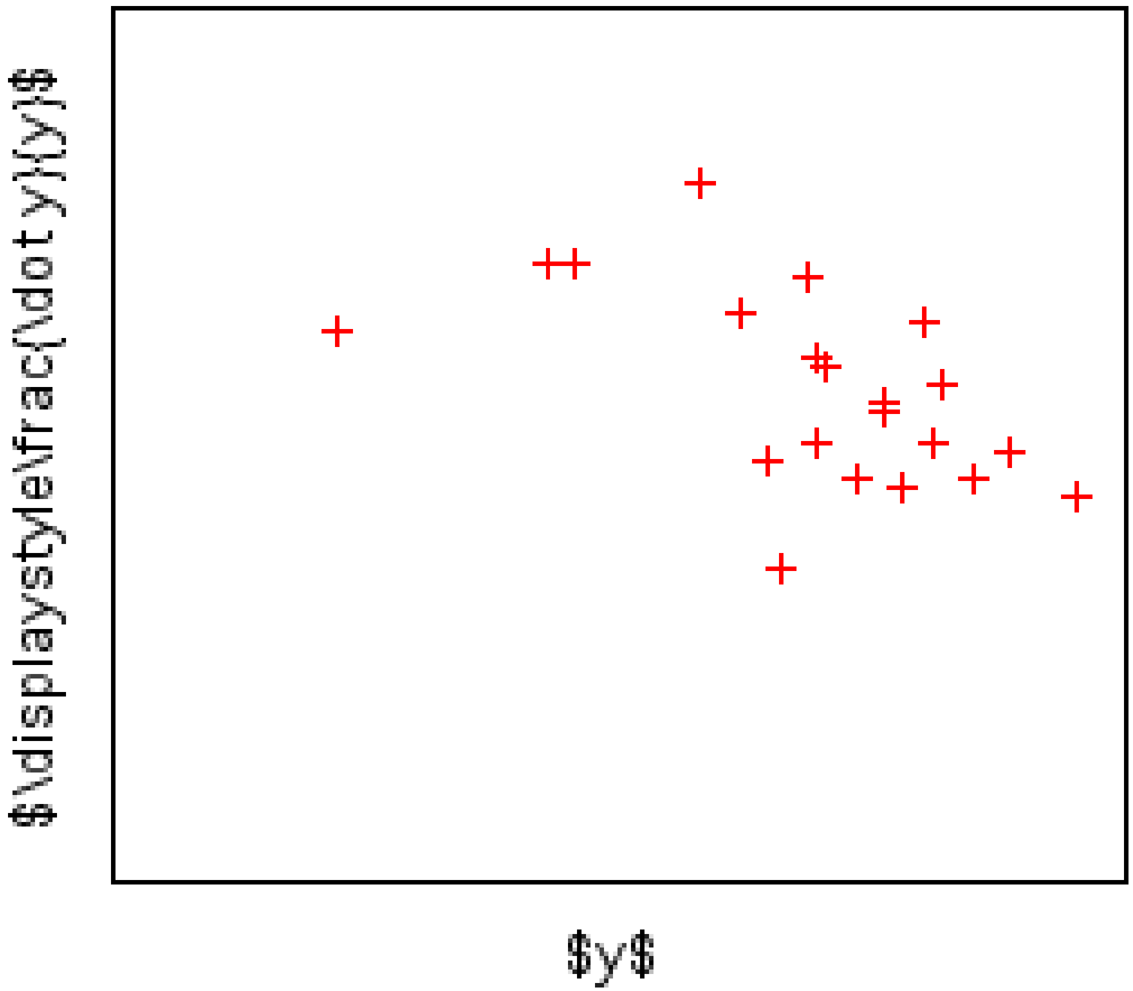
Empirical Evidence on Convergence

- Method: compare income per capita to growth rates, "poor grows faster" implies downward slope of plot
- Baumol showed convergence in OECD for 1870–1994, Fig. 3.3 (see also Fig. 3.4)
- A similar exercise works well for OECD 1960–90, but not for the world.
- Why not?
  - Jones, Fig. 3.2, suggests steady state growth model explains relative income patterns well.
  - Most differences are due to differences in savings rate, population growth, and "technology." (Estimated TFP tends to rise with income.)

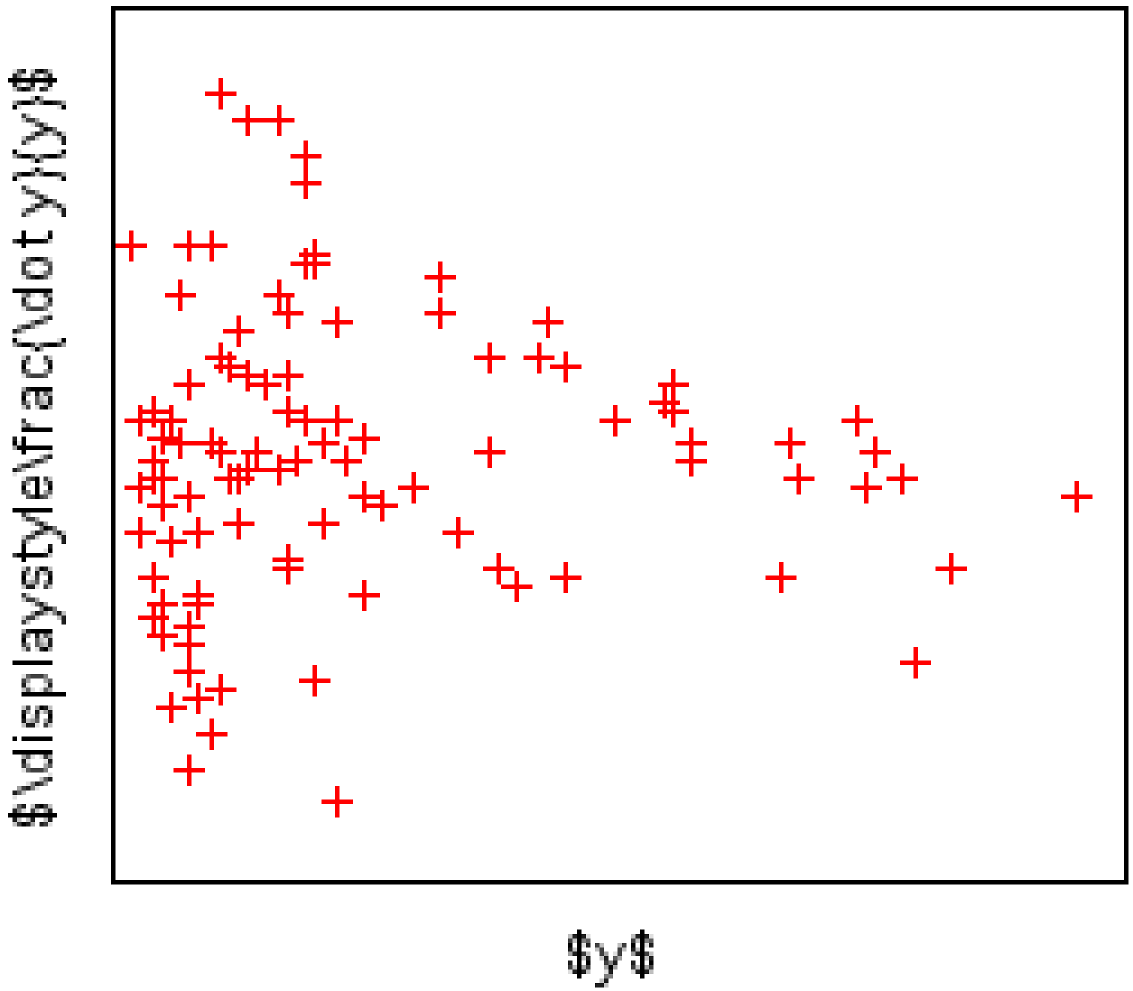
OECD Convergence 1960



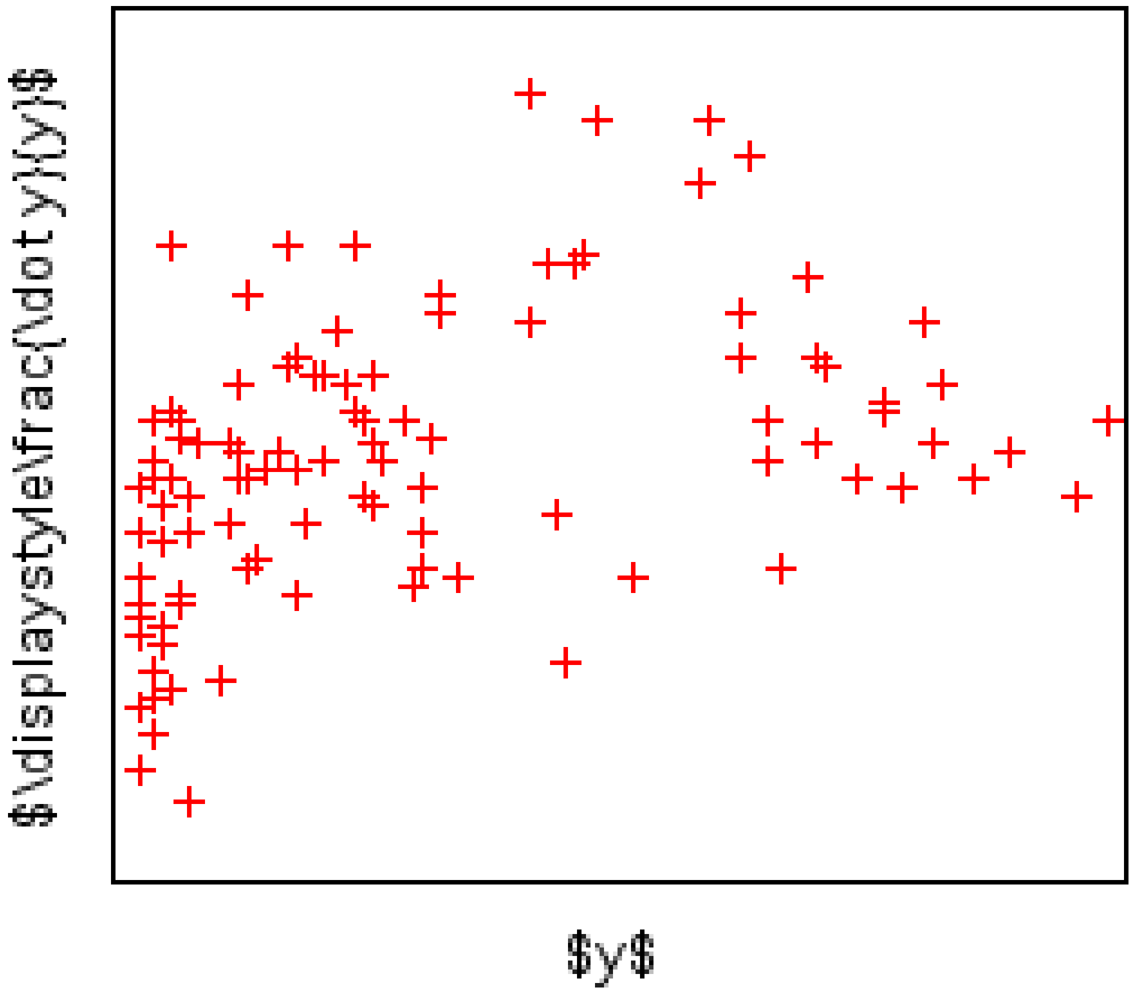
OECD Convergence 1990



World Non-convergence 1960



World Non-convergence 1990



A Standard of Comparison: The U.S.

- What does *typical long term growth* look like? Let's look at the best example of a mature economy, the U.S.'s.
  - growth takeoff around 1850 (like Japan)
  - no war damage to capital stock during the modern period (unique)
- The *real rate of return* to capital shows no upward or downward trend in the U.S., 1850–1990.
- The shares of capital and labor in income ($\frac{wL}{Y}$ *vs.* $\frac{rK}{Y}$) show no trend.
- This pattern of economic progress is called *balanced growth*, and it is consistent with Solow's model.
- *But* the average rate of output growth per worker is positive and seems constant over the period (Solow predicts zero).

Empirical Patterns of Growth: Other Patterns

These facts can't be analyzed in the Solow model because it has only one country.

- Growth in output is positively related to trade growth.
- Skilled and unskilled workers migrate from poor to rich countries. The former is a paradox: skills are "human capital" and poor countries have little capital. Returns to capital should be high in poor countries.
- In fact, until recently, capital of all kinds has tended to flow out of poor countries.
  - Conservatives argue that this is due to poor capital markets, and unstable and ineffective governments.

Partial explanations

For each of the following camparisons, ask

- which has the theoretical advantage? then
- which has had higher growth?

Resources: Argentina, Russia *v.* Japan The natural resources of Argentina or Russia are vastly greater than those of Japan in the 20th century.

Capitalism: "Tigers" *v.* USA Japan and the Asian "tigers" probably have more corruption, are definitely far less market-oriented (especially in the capital and labor markets), and have far more intrusive and obstructive bureaucracies than that of the U.S.

Technology: North *v.* South Advanced technology *is* connected with high growth rates. But it is still a puzzle why technology is not easily transferred to emerging economies.

Current research in economic growth theory, as well as in international busines and in management of technology, addresses these issues.

The Solow residual

- Recall that a stylized fact of U.S. economic growth is that per-worker output has grown at 1.5%–2% per year for around 150 years.
- Solow's model predicts 0. The difference is called "the (Solow) residual," which is a good name for it because it doesn't imply an explanation. Solow himself called it "a measure of our ignorance" (about the details of the mechanism of economic growth).
- Of course we know that in the modern age "technological progress" is occurring around us all the time.
- But how should *economics* model it? Can its rate be explained by economics, or is it some sort of arbitrary natural law like population growth from the point of view of economics?
- Economists have chosen to represent technological progress by *productivity improvements*.
  - This summary figure ignores actual innovations in favor of measuring economic effects. Compare the *cost function* for production.

Estimating the Production Function

- We assume a Cobb-Douglas production function:

$$Y = F(K, L) = AK^{\alpha} L^{1-\alpha}.$$

- A and α can be estimated using the linear regression specification:

$$\ln Y = \ln A + \alpha \ln K + (1 - \alpha) \ln L + \epsilon.$$

- In country-specific analysis, we use time-series analysis. A convenient specification is *Hicks-neutral* technological progress with $A = A_0 e^{\lambda t}$. Then the regression equation is

$$\ln Y = \ln A_0 + \lambda t + \alpha \ln K + (1 - \alpha) \ln L + \epsilon,$$

  which is very convenient.

  – Here the variables are *dated*. With technological progress, the production function is dated, and changes over time: $Y_t = F_t(K_t, L_t)$.

- In the "convergence" analysis, we use a cross-section estimation assuming all the countries have the same technology. (We could also use panel data, and assume common λ as well.)

Including Country-Specific Technology

- The next step in improving fit is to adjust country by country for differences in technology, using panel data. A *panel data* set is one where you have a time series for each individual, with each variable measured in the same way for all individuals.

- We simply estimate a separate coefficient A (total factor productivity) for each country i:

$$\ln Y = \ln A_i + \alpha \ln K + (1 - \alpha) \ln L + \epsilon$$

  α is still common across countries, but we allow for different levels of productivity.

Including Country-Specific Characteristics

The next step in improving fit is to adjust country by country for differences in saving behavior and human capital.

- Savings and fertility behavior is simple: just find out the gross savings rate s and labor force growth rate n for each country. These country-specific characteristics are just historical averages.
- It is convenient to use the Cobb-Douglas production function $F(K, L) = AK^{\alpha}L^{1-\alpha}$. We assume a common level of $\alpha = 0.3$, but allow each country to have a different A.
  - In a competitive economy, $\frac{rK}{Y} = \alpha$ and $\frac{wL}{Y} = 1 - \alpha$, so α is easy to estimate.
- Finally we adjust for "human capital" by assuming human capital can be measured by education u, and putting $\hat{L} = e^{\phi u}L$, and then $Y = F(K, \hat{L})$.

Production Function with Human Capital

	act	pred	s	u	n	$\hat{A}_{90}$
U.S.A.	1.00	1.00	0.210	11.8	0.009	1.00
W. Germany	0.80	0.83	0.245	8.5	0.003	1.02
Japan	0.61	0.71	0.338	8.5	0.006	0.76
France	0.82	0.85	0.252	6.5	0.005	1.28
U.K.	0.73	0.76	0.171	8.7	0.002	1.10

act and **pred** are the *actual* and *predicted* levels of output per worker relative to the U.S., **s** is the gross saving rate, **n** the labor force growth rate, and $\hat{A}_{90}$ the estimated productivity for 1990.

The figures for Japan and France are interesting:

- France's technology parameter is extremely high, much higher than the U.S., which is not plausible.

- Japan's is much too low.

Human Capital

- *Human capital* is the result of investment in skills and knowledge of workers. It is *capital* because it persists for the working life of the worker.
  - It comes in two types: *general* and *firm-specific*. This distinction is extremely important in labor economics, but we will ignore it.
- In growth theory, human capital is most conveniently represented as a labor-enhancing factor: $F(K, A(E)L)$, where the variable E is the level of training, or more generally, *education*.
- Depending on assumptions, convergence to *steady state balanced growth* may occur, leading to a constant average level of education, and constant levels of all per-capita variables (though at higher levels than without education).
- Alternatively, the "education stock" may grow at a constant rate like general labor-enhancing technological progress, and the results are as described earlier. Unfortunately, this seems implausible.

Solow Model with Technological Progress

- In the Solow model, the production function has one output "goods," and two inputs "labor" and "capital." The most general way to model technological progress is to make the production function *time-dependent*: $F(K, L, t)$. We generally assume F is CRTS at each time t separately.
- A simpler way of modeling technological progress is to treat it as a *productivity enhancement*. With three variables involved, we can enhance all of them: $\alpha(t)F(\beta(t)K, \gamma(t)L)$. Again, we assume F itself is CRTS.
- Depending on which factors α, β, and/or γ may be different from 1, we have

parameter	enhances	neutral
$\alpha > 1$	total factor productivity	Hicks
$\beta > 1$	capital	Solow
$\gamma > 1$	labor	Harrod

Representing Technological Progress

Basically, technology is represented as “enhanced” inputs or outputs, by multiplying them by a factor we may call “productivity.” This factor increases over time.

- To derive a tractable *characteristic equation* for technological progress, it is easiest to assume *labor-enhancing* progress, with a production function of the form $F(K, A(t)L)$. We also call this *Harrod-neutral* progress.
  - Define *effective labor* by $\tilde{L} \equiv AL$.
  - Define $\tilde{y} \equiv \frac{Y}{AL}$ and $\tilde{k} \equiv \frac{K}{AL}$.
  - Assume A grows at rate $g > 0$ (*i.e.*, $A(t) = A_0 e^{gt}$). Then $\tilde{L} = A_0 e^{gt} L_0 e^{nt} = A_0 L_0 e^{(n+g)t}$ grows at rate $n + g$.
- The steady state is defined by $\dot{\tilde{k}} = 0$, where the characteristic equation is

$$\dot{\tilde{k}} = sf(\tilde{k}) - (n + g + d)\tilde{k}.$$

Balanced Growth with Technological Progress

- Since the saving function hasn't changed and the capital thinning function is still linear, with the Inada conditions the analysis is the same as in the basic Solow model: there is a stable steady state $\tilde{k}^* > 0$.

- The comparative statics of $\tilde{k}^*$ is the same as before with respect to all variables, and g has the same properties as n or d.

- But note that $k = A\tilde{k}$ and $y = A\tilde{y}$, so $y^* = \frac{Y}{L}$ and $k^* = \frac{K}{L}$ grow at rate $\frac{\dot{A}}{A} = g > 0$ in the steady state where $\tilde{k}^*$ does not grow.

- Notice that by redefining the state variable from k to $\tilde{k}$ (and assuming the "right" kind of technological progress), we can easily extend the analysis in this way.

Other Representations of Technology

- Growth accounting (presented next) is based on *Hicks-neutral* technological progress, using a production function of the form $A(t)F(K, L)$ where A is the technology index and F is a constant returns to scale production function.
- For completeness, we can define *Solow-neutrality*, of the form $F(A(t)K, L)$. This is a plausible representation of the kind of technological progress produced by "R&D."
- Of course, we can combine all three in theory. That might be the most accurate representation of reality, but in practice each kind of progress is useful in different applications.

Endogenous Technological Progress

- We won't be able to give a full growth model with endogenous technological progress; it's too complex.
- First, note that the macro models are missing a very important idea: physical capital becomes obsolete as well as breaking down.
- We could model this as a higher rate of depreciation d.

  But: d depends on $\dot{A}$ (the change in "technology level").
- Alternatively, we could use a "vintage" model, in which only the capital bought at time t has productivity $A(t)$.

  But: we either must associate labor with specific vintages of capital (complicated!), or we must use a model with Solow-neutral technical progress—which neither provides a base for simple growth accounting, nor is easily adapted to the Solow method for analyzing growth models.

Multi-Sector Models

- Simplest way to make d endogenous is a two-sector model, with one sector "producing" technology (R&D).
- This is dynamic: current investment constrains the future "stock" of technology (it must grow).
- It is irreversible.
- Compared to Solow model, basic result (a steady-state balanced growth path exists) is the same.
  - The Japanese economist *Hirofumi Uzawa* is most famous for this analysis.
- The math is more complicated. Stability of steady-state balanced growth path depends on the parameter comparing the production functions for physical capital and technology.

Growth Accounting

- “Growth accounting” is a simple idea: decompose the overall rate of economic growth according to its sources.
  - Factor growth
  - Technological growth
- In steady state balanced growth, it is trivial.
  - All factors grow at the same rate (which is the labor force growth rate).
  - The difference between the rate of growth of GDP and the common factor growth rate is the *residual*, also called *total factor productivity (TFP) growth*.
  - TFP growth is interpreted as technological progress.

Growth Accounting, cont.

- Outside of steady state balanced growth, factors grow at different rates.
  - Use econometrics to estimate the production function.
  - Use the production function to estimate the contribution of each factor.
  - Any residual is TFP growth (by *definition*).
- TFP can be included in any production function regression simply by multiplying by $e^{\lambda t}$ (*Hicks-neutral technical progress)*:

$$\hat{F}(K, L, t) = e^{\lambda t} F(K, L),$$

  $\hat{F}$ includes the effects of time, *i.e.*, technological progress. With a Cobb-Douglas function, linearizing via the logarithm gives:

$$\ln Y = \ln A + \alpha \ln K + (1 - \alpha) \ln L + \lambda t + \epsilon.$$

Growth Accounting: Example

- Assume a Cobb-Douglas production function with $\alpha = .25$: $Y = K^{.25}L^{.75}$.
- In one year, labor force grew by 2%, capital stock by 4%, and output by 3%.
- Then, comparing to the base year we have

$$\begin{aligned}(1.04K)^{.25}(1.02L)^{.75} &= 1.04^{.25}1.02^{.75}K^{.25}L^{.75}\\ &= 1.04^{.25}1.02^{.75}Y = \hat{Y},\end{aligned}$$

so $\lambda = 1.03 - 1.04^{.25}1.02^{.75} = 0.005036$ where λ stands for TFP growth, and is approximately 0.5%.

Growth Accounting Results

U.S.

Assume Cobb-Douglas production function with $\alpha = 1/3$. Rows do not add due to rounding. Numbers are percentage growth rates of GDP, except the numbers in parentheses are factor growth rates.

		Contribution to GDP			
Period	GDP	Capital	Labor	TFP	GDP/L
1960–70	4.0	0.8 (2.4)	1.2 (1.8)	1.9	2.2
1970–80	2.7	0.9 (2.7)	1.5 (2.3)	0.2	0.4
1980–90	2.6	0.8 (2.4)	0.7 (1.1)	1.0	1.5
1960–90	3.1	0.9 (2.7)	1.2 (1.8)	1.1	1.4

TFP and Recession in Japan

- Estimated production function: $Y = AK_o^{0.1321}K_h^{0.1267}K_s^{0.0522}L^{0.689}$

	Y	K_o	K_h	K_s	L	TFP
1975-1999	3.09	0.75	1.18	0.65	0.65	-0.14
1975-1980	4.24	1.24	0.79	1.02	1.20	0.01
1981-1985	3.50	0.72	1.53	0.71	1.06	-0.56
1986-1990	5.06	1.02	1.71	1.06	1.22	0.09
1991-1995	1.49	0.27	0.88	-0.10	0.15	0.28
1996-1999	0.40	0.34	1.01	0.45	-0.79	-0.66

Homework Submissions

1. Submit your homework *by email to*

   `"Economic Dynamics" <turnbull@sk.tsukuba.ac.jp>`

   The `Subject:` should be `Dynamics (FH27041) HW #1`. (For assignments #2, #3, and so on, adjust the homework number.)

2. Without the class number and the homework assignment in hankaku romaji, your email may get lost. Use the class number above, even if you are registered according to a different code.

3. Your email must contain your *name* and *student ID number*.

Homework Format

5. For simple answers, I *strongly* prefer *plain text* or TEX to *Word documents* and *HTML*. In plain text, you may write subscripts using programming notation (*i.e.*, X_t becomes `X[t]`), and superscripts using the caret (*i.e.*, X^t becomes `X^t`) or double-star (X^t becomes `X**t`).

Writing homework answers

This page is under development. More information will be added as I notice issues.

Many of the tasks assigned in homework are expressed using idioms specific to this class.

solve Also **give a solution** or **derive**. You *must* show your work. Obvious calculations of common operations, such as the $6 \times 5 \times 4 \times 3 \times 2 \times 1$ in $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ may be omitted, but even slightly more complex operations such as ${}_6C_3 = \dfrac{6!}{3!3!} = 20$ should be written out.

discuss Most important, relate the computation to the real problem in economics (or physics or biology for some of the "toy" examples). Especially mention anything paradoxical, surprising, or extreme about the interpretation of the result in context of the real problem.

compare Like **discuss**, but more specific: you should use statements of the form "*this* is the same as *that*," "*this* is different from *that*," and (best) "*this* is similar to *that*, except ...."

show ... is Often you need to transform one of the expressions to the other. You must show your work, not just "$expr_1 = expr_2$ (same!)"

notation You may define your own notation whenever convenient. For example, in HW#2, Q#3 you're asked to compare δ in Q#1 to δ in Q#2. This gets confusing and long winded (*i.e.*, because you write "δ of Problem 1" over and over again). It may be useful to rewrite one of the results (in this case, that of Q#2) by substituting γ for δ everywhere.

Homework 1: October 23, 2020

Consider the *linear production function*:

$$F(K, L) = rK + wL.$$

1. Does the linear production function have *constant returns to scale*? Prove your answer.

2. Is the linear production function *neoclassical*? Show your work, and be careful about treatment of equalities!

3. The linear production function is not often used (at least not in this form) in growth theory. Why is this function uninteresting to economists? (Thought question: That is, any guess is a good guess, but please do try to answer this question.)

4. Why do you think I chose the coefficients r and w in the equation defining the linear production function? (Another "thought question.")

Homework 2: October 23, 2020

This exercise concerns *dated commodities* and *dated technology (production functions)*.

1. Let $F_t(K, L)$ denote Toyota Company's production function for cars in year t.
   (a) What does $F_{2006}(K, L) < F_{1996}(K, L)$ mean?
   (b) Do you think the inequality in (a) is a historical fact? Explain why or why not.
   (c) What does $F_{2006}(K_{2006}, L_{2006}) > F_{1996}(K_{1996}, L_{1996})$ mean?
   (d) Ignoring historical facts about Toyota, is it logically possible for both (a) and (c) to be true? Explain why or why not.

Homework 3: October 23, 2020

1. Consider the Cobb-Douglas production function with *Hicks-neutral* technological progress:

$$F(K, L, \lambda) = A_0 e^{\lambda t} K^{\alpha} L^{1-\alpha}.$$

   (a) Show that for appropriate constants B_0 and β the Hicks-neutral technological progress can be expressed as a production function with *Solow-neutral* technological progress:

$$F(K, L, \beta) = B_0 (e^{\beta t} K)^{\alpha} L^{1-\alpha}.$$

   (b) Show that for the appropriate constants C_0 and γ the Hicks-neutral technological progress can be expressed as a production function with *Harrod-neutral* technological progress:

$$F(K, L, \gamma) = C_0 K^{\alpha} (e^{\gamma t} L)^{1-\alpha}.$$

2. Explain why Solow-neutral technological progress is also called *capital-enhancing* technological progress. Use the equation in your answer.

3. Explain why Harrod-neutral technological progress is also called *labor-enhancing* technological progress. Use the equation in your answer.

4. Explain why the method used to adapt Solow's analysis to Harrod-neutral technological progress will *not* work for Solow-neutral technological progress. *Hint: the answer I have in mind uses the idea of* exogenous *and* endogenous *variables. There are probably other good ways to explain it, however.*