Economic Dynamics

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Lecture 1: October 5, 2018

Abstract

Introduction to Economic Dynamics, the course, the instructor, and the field. Second period continues with discussion of mathematical foundations of dynamic analysis.

Course Description

This course introduces dynamic analysis in economics. It presumes familiarity with intermediate microeconomics, using optimization theory to explain consumer and firm behavior as a foundation for analysis of economic interactions, primarily via markets.

It is a pure lecture course, with evaluation based on examinations and out-of-class problem sets. By request of the curriculum committee, the primary language of instruction is **English**. Lectures and course materials will be in **English**. Class discussion and questions may be in English or Japanese at your convenience. Out-of-class assignments should be written in English, but may be written in Japanese *with prior approval from the instructor*. Expect substantial delays in evaluation of work written in Japanese *without approval.*

Evaluation

Grading will be based on a final examination, a midterm examination, problem sets, and class participation, in that order (weighted approximately 4:3:2:1). Examinations may be answered in English or Japanese. "Class participation" means asking questions, whether to clarify presented material or to ask about applications of class material to "real-world" problems. Experience shows that my exam grading is not biased by my feelings about students, but when assigning course grades you will *not* get a worse grade if I can associate your name with a face. You may get a better grade, specifically if you are at or near the cutoff.

Primary Course Resources

- Class notes should be downloaded from the course home page: `http://turnbull.sk.tsukuba.ac.jp/Teach/Dynamics/`.
- **Optional:** Ronald Shone [2001], *Introduction to Economic Dynamics*, Cambridge, UK: Cambridge U. Press is **optional**. The bookstore won't handle foreign texts unless at least 75 students are expected. It can be ordered from Amazon but it's pretty expensive.

Secondary Course Resources: Background

- Nishimura, Kazuo [1995]. *Mikurokeizaigaku Nyuumon*, 2d ed. (Japanese). Tokyo: Iwanami Shoten. (This is a good textbook in Japanese.)
- Varian, Hal R. [2014]. *Intermediate Microeconomics: A Modern Approach*, 9th ed., New York: Norton. (This is the best textbook in English. For this class, any edition will do.)
- Bergstrom, Theodore, and Hal R. Varian [2014]. *Workouts in Intermediate Microeconomics*. (Companion to Varian [2014].)
- Hirsch, Morris W., Stephen Smale, and Robert L. Devaney [2013]. *Differential Equations, Dynamical Systems, and an Introduction to Chaos*. Waltham, MA: Academic Press. (For more information about the mathematics of dynamical systems. Not easy.)
- Intriligator, Michael. *Mathematical Optimization and Economic Theory*. (For more information about optimization and control theory. Not easy.)

Secondary Course Resources: Theory

- Gandolfo, Giancarlo [2009]. *Economic Dynamics*, 4th ed., Berlin: Springer. (Currently the encyclopedic resource on economic dynamics.)
- Lucas, Robert and Nancy L. Stokey. *Recursive Methods in Economic Dynamics*.
- Shone, Ronald [2002]. *Economic Dynamics: Phase Diagrams and their Economic Application*, 2nd ed., Cambridge, UK: Cambridge U. Press. This is the long version on which *Introduction to Economic Dynamics* is based.
- Stokey, Nancy L. *The Economics of Inaction*. (A text on the most recent advances in stochastic dynamics.)

Secondary Course Resources: Applications

- Arthur, W. Brian [1994]. *Increasing Returns and Path Dependence in the Economy*, Ann Arbor, MI: U. of Michigan Press.
- Axelrod, Robert [1997]. *The Complexity of Cooperation: Agent-Based Models of Competition and Collaboration*, Princeton, NJ: Princeton U. Press.
- Dasgupta, Partha, and Geoffrey Heal. *Economic Theory and Exhaustible Resources.*
- Jones, Charles. *Economic Growth Theory.*
- Kleinberg, Jon & David Easley. *Networks, Crowds, and Markets.*
- Resnick, Mitchel [1994]. *Turtles, Termites, and Traffic Jams: Explorations in Massively Parallel Microworlds*, Cambridge, MA: MIT Press.

Instructor information

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Course information

Economic Dynamics (FH27041)	
Prerequisites	Microeconomics and elementary calculus.
Time & room	Fri 3&4, 3L202
Home page	`http://turnbull.sk.tsukuba.ac.jp/Teach/Dynamics/`
Lecture notes	Linked from course home page
Contact email	`dynamics-help@turnbull.sk.tsukuba.ac.jp`
Homework submission	`dynamics-hw@turnbull.sk.tsukuba.ac.jp`

The Dynamic World

What is the subject of economic dynamics? While many economic problems have a "generic static solution" that can be applied repeatedly in several periods, in many problems linkage must be considered over time. Decisions made now create constraints or opportunities for decisions in future periods.

Dynamic analysis is important:

- "The uncertainty of life itself casts a shadow on every business transaction into which time enters." (Irving Fisher, *The Theory of Interest*, Ch. IX, p. 216.)
- "Time ... is the centre of the chief difficulty of almost every economic problem." (Alfred Marshall *Principles of Economics*, Preface to the First Edition.)

Is Comparative Statics Enough?

- Elementary microeconomics compares *equilibrium states*, using the technique of *comparative statics*.

- Before and after? Maybe, maybe not.

  Comparative statics cannot be thought of as "before and after." Two different equilibria are based on contemporary conditions. Returning to the putative "before" equilibrium is trivial.

  In realistic problems, however, the conditions *evolve* over time, and reversing the process is not possible. (Arthur [1994] on "path dependence", Gandolfo [2009] on "irreversibility in dissipative nonlinear systems and chaotic systems".) Resetting policy to the original state is *not* enough to reset all environmental conditions.

- Thus continuing change is central, and not only must evolution of physical conditions be considered but persistent changes in the *expectations* held by market participants.

Anticipation and Demand

- Several times in the last decades the government of Japan has increased the consumption tax rate (including the introduction of the tax, which can be thought of as raising the rate from 0% to 3%).
- The static equilibrium analysis shows that consumers perceive this as a price increase (*zei-komi kakaku*) and firms as a price decrease (*zei-nuki kakaku*). The burden is shared according to the ratio of price elasticities of supply and demand.
- However, when the consumers and firms anticipate the tax increase, there is an additional effect of anticipatory purchasing: consumers change their plans to buy large durables (houses, cars, refrigerators) to complete purchase *before* the tax increase. That is, the announcement induces a rightward shift in demand in the pre-tax-increase period (*kake-komi juyou*), and a leftward shift in the post-tax-increase period.
- This mitigates the decrease in sales over the long term, and reduces the tax revenue, compared to the static analysis.

Complex Dynamics

- The “classical” dynamic models are aggregate models. They treat markets (with hundreds or thousands of traders reduced to price and quantity), or populations (reduced to the number of members), or whole national economies (reduced to GDP and population).
  - Mathematically we say they are of *low dimension*, that is, they have few variables.
  - They also tend to be either monotonic or cyclic according to a sine wave.

  These two properties make them “simple”.

- “Complex” dynamics arise when either property fails.
  - With low-dimensional nonlinear systems, *chaos* can arise.
  - In systems with many interrelated variables (such as “agent simulations”), the dimensionality itself creates complexity as changes ripple back and forth “across” the “population” (from one agent’s variables to other agents’ variables).

"Classical" Dynamic Topics in Economics

- *Economic growth theory* can help us to understand the context of Japan's current economic discomfort better – and predict that Korea, Taiwan, and China will experience similar discomfort.
- *Technological innovation* is at the core of modern growth theory; we'll take a look at dynamics of innovation.
- Do *limitations on natural resources* necessarily imply "limits to growth"? We will look at the economics of exhaustible resources, like oil, and renewable ones, like fish.
- Why are markets so *volatile*, with a persistent but irregular business cycles and financial bubbles that end in "meltdowns"? *Stability analysis* can distinguish inherent instability from external randomness.
- The theory of *pricing of derivative assets* necessarily involves time (as Irving Fisher said).

Basic Ideas of Economic Dynamics

- Dynamics studies an evolving process of change.
- Dynamics often involves irreversibilities.
- Commodities consumed at different *dates* are considered to be different commodities. *E.g.*, "storage" is *production*. Consumers are considered to have *time preference*, or *discounting*.
- Dynamic processes can be very complex, since tomorrow's outcome of an action today may be outweighed by later outcomes depending on that action. Irreversibility makes it possible to consider simple "endgames", and analysis works back from there.
- Dynamic optimization may also focuse on *steady states* where the decision-maker's problem does not change. *I.e.*, the decision in each period recreates the original conditions for the next one.

Microeconomics and Dynamics

- Economic dynamics is a branch of microeconomics. Microeconomics is a science:
  - quantitative, and
  - logically correct.
- Microeconomics guides "social engineering,"
  - as chemistry guides chemical engineering.
- Rigor (logical correctness) is costly, requiring
  - abstraction (modeling), and
  - explicit statements of the relationships that are important.
  - *Ad hoc* accounting for "other factors" is unscientific.
  - Instead, change the model, and re-solve from the beginning.

Review of Microeconomic Modeling

- Abstraction eliminates “unimportant” details.
- Analysis is easier and more transparent.
  - Classroom models emphasize “easy to study.”
  - Policy models emphasize computation of accurate predictions.
  - Theoretical models need high degrees of abstraction.
- Loss of realism is a necessary cost,
  - of complexity of individual decision-making and interaction.
  - Our models are less robust than those used by engineers,
  - and less emotionally satisfying (to laymen and to many “political economists”) than a careful verbal analysis.
- Are they realistic enough?

A Definition of Microeconomics

Microeconomics: the social science that studies interactions among allocations of scarce resources to competing ends by rational agents.

science Science is organized, quantitive study using *models*. "Quantitative" means measurement and mathematics.

model A *model* simply states that certain facts and relationships are important, and others will be ignored. Discussing other facts and relationships *violates* the model.

social science A *social* science is one that deals with *interactions* within groups, of animals, organizations, or even machines. Explicit description of interactions is why microeconomics is "micro". Macroeconomics, on the other hand, concentrates on overall flows of resources rather than decisions.

scarce resources A *resource* is anything that might ever be useful to somebody. A resource is *scarce* when someone would be able to use a little bit more of it. A resource is *free* if nobody anywhere has any use whatsoever for more of it.

competing ends An *end* is a use, goal, or purpose. Two ends *compete* when they both could use the same scarce resource.

allocation *Allocation* is simply a fancy word that means to make a decision. In particular it refers to a decision about how to use a particular resource.

rational agent *Agent* simply means "decision maker." *Rational* simply means "purposeful" or "goal-oriented."

Dimension of variables

- In formulating and interpreting a dynamic model, it is important to keep track of the dimensions (units) of various quantities.
- Algebra "doesn't care" what variables and numbers you combine, but in economics most combinations don't make sense.
- The most important task of the economic theorist is checking that the model equations make sense, or revising them so that they do make sense.
- The classic saying is that "you can't add apples and oranges." That is true, in one sense, but that would make it impossible to do macroeconomics. The macroeconomist revises the model to "count" apples and oranges in terms of their *values*, and adds those up to get *GDP*.
- In the same way, microeconomists summarize all of the expenditures of production in a *cost function*, or give different items a *quality index* without explaining why one is "better" than another.

Stock *vs.* flow

- The most important dimensional distinction in economic dynamics is that of *stock vs. flow.*
- In economics and accounting, the *stock* of a good is the amount that exists at a given time. A stock has a dimension of physical units: number of apples, kilos of rice, carats of diamonds, and so on.
- A *flow* is a *rate of change* in a stock. Therefore it is measured as a ratio: physical units per time unit.
  - It is *not* the change in a stock. It needs to be multiplied by time; if the flow exists only for an instant, the stock doesn't change. The longer the flow exists, the bigger the change in stock.

Stock *vs.* flow: example

- For example, it makes sense to say one has a *wealth* of 10,000,000 yen. It does *not* make sense to say one has an *income* of 10,000,000 yen, because it means very different things depending on the unit of time.
- A person with an income of 10,000,000 yen *per month* is *very* fortunate.
- An income of 10,000,000 yen *per year* is pretty good (but well within reach: a senior professor acting as Dean of Shako makes about that much).
- An income of 10,000,000 yen *per lifetime* is extreme poverty (based on an average lifetime of 75 years).

Higher-order dimensions

- In mechanics, we are familiar with higher-order units. For example, *acceleration* has units "distance per time squared."
- Such units are less common in economics, but they occasionally show up in highly dynamic fields such as finance and macroeconomics.

Mathematics Used in Economic Dynamics

- The mathematics used in economic dynamic analysis ranges from very basic to horribly advanced. You don't need to worry about the advanced stuff, but it's there if you like math.
- Basic calculus: dynamics is fundamentally concerned with the interplay of *level* and *rate*.
  - In differential calculus, we look at a sequence of levels, or *level as a function of time* ($x(t)$), and derive the *rate of change* as the derivative:

$$\dot{x}(t) = \frac{d}{dt}x(t).$$

  - In integral calculus, we look at a sequence of *rates* ($y(t)$) and derive the level as a function of time:

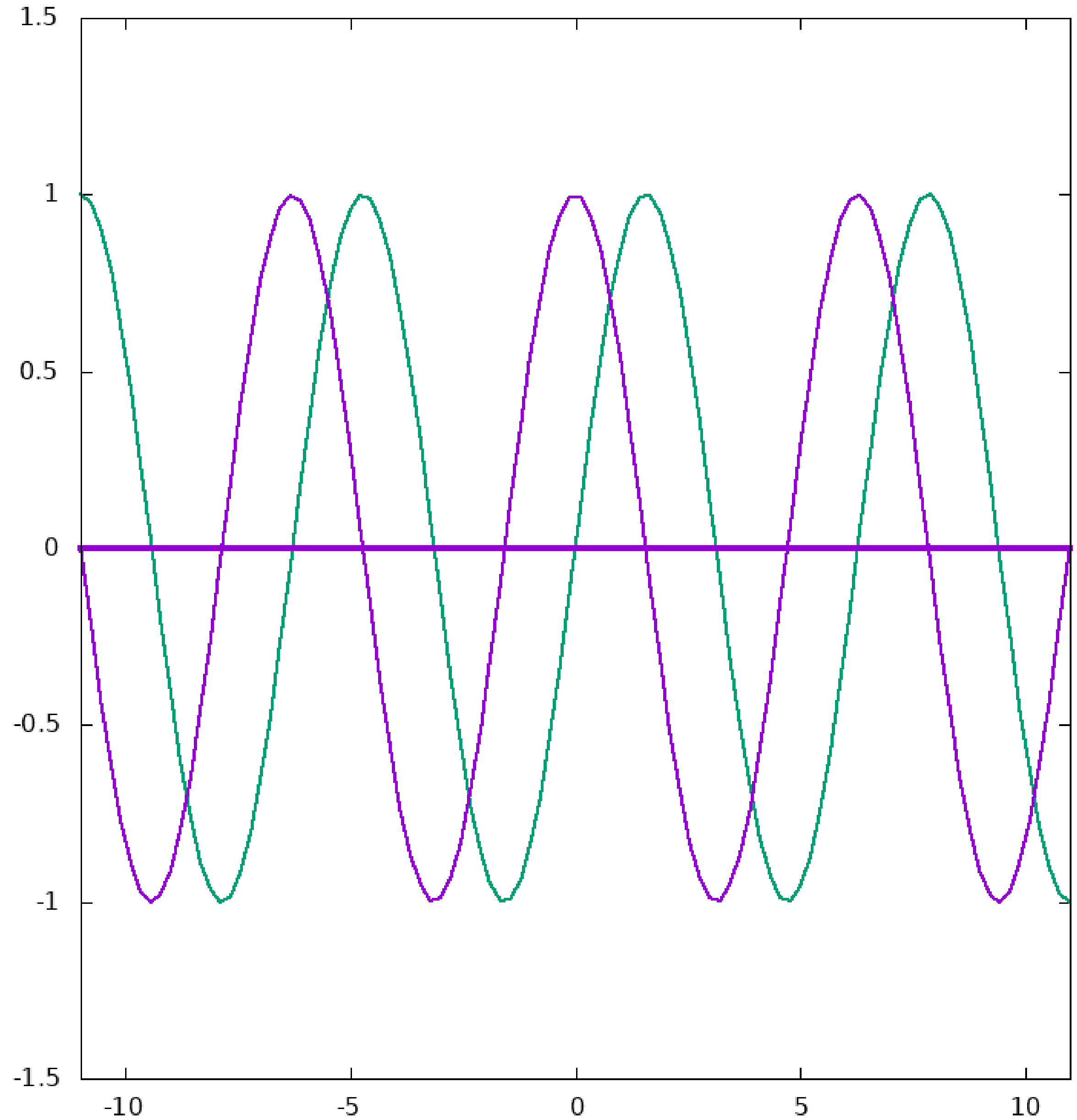
$$Y(t) = \int_{t_0}^{t_1} y(t)dt + Y(t_0).$$

- I'll talk about the harder stuff when we get there.

Differential equations

- A *differential equation* is a mathematical expression of a constraint involving certain variable quantities and their derivatives.
  - A differential equation is often called a *law of motion*, but they arise in other contexts, such as determining the shape of a chain hanging from two points.
  - A set of differential equations relating a specific set of variables is called a *system* of differential equations.
- Differential equations can be graphed in many ways: time series, phase diagrams, vector fields, and many others.

Time series: sin x *vs.* cos x

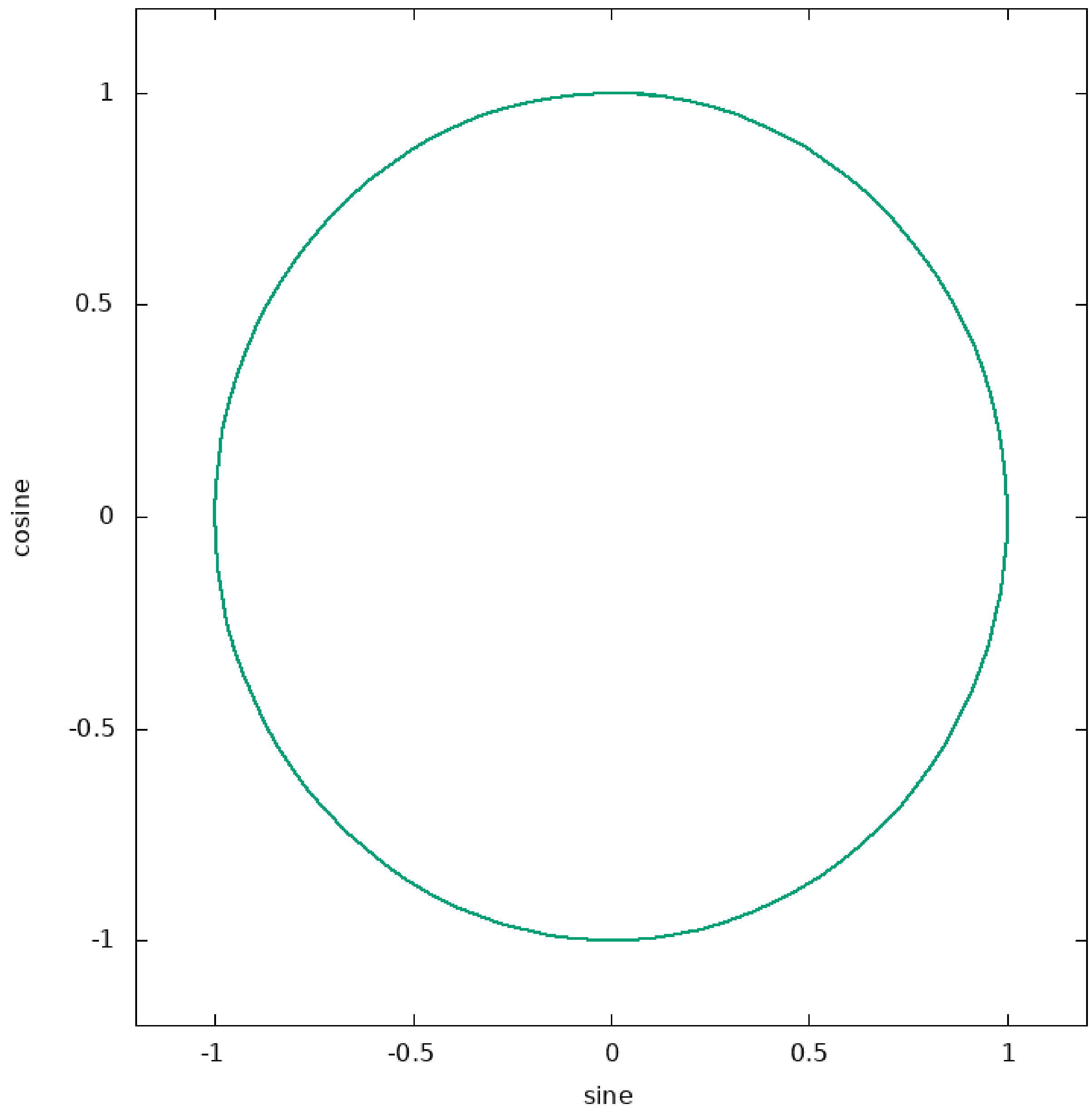


Sine and cosine chase each other (high and low) across the graph.

Phase diagrams

- In economics, the most useful graph is the *phase diagram*, which graphs the combinations of values of two variables that occur simultaneously. Phase diagrams do not show *when* the combinations occur, only that they do at some point.
  - The *parameter* (in dynamics, the *time variable*) is implicit.
  - Because of *continuity* the phase diagram shows the order of values. Thus the graph is often called an *orbit*.
  - It doesn't show direction of time flow.
- For example, you saw the graph of $\sin x$ and $\cos x$ against x. This is not a phase diagram. However, you can see that they are "chasing" each other. The exact relationship is displayed on a phase diagram of $\sin x$ against $\cos x$, and it is the *circle* $\sin^2 x + \cos^2 x = 1$.

Phase diagram: sin x *vs.* cos x



Sine and cosine chase each other (high and low) across the graph.

Qualitative properties of differential equations

- In economics, we often do not have very precise information about differential equations we use. *E.g.*, $dx/dt = f(x, t)$, with some marginal conditions on f.
- Even if we do, we may not be able to solve to get an explicit function $x(t)$.
- In looking at *economic growth theory*, we will focus on one set of qualitative properties: those involving the steady state, x such that $\dot{x} = 0$. *Market stability analysis* also revolves around steady states, with the special property that they should correspond to the supply-demand equilibrium.

Classifying differential equations

- If all of the unknown quantities are functions of one of the quantities, all of the derivatives may be reduced to ordinary derivatives, and the equation is called an *ordinary differential equation.*
  - The single quantity is called the *parameter.* In dynamics, the parameter is interpreted as *time.*
  - Otherwise, partial derivatives are involved, and the equation is called a *partial differential equation.*
  - If a system of differential equations contains any partial differential equations, the whole system is classed as a system of partial differential equations.
- The *order* of a system of differential equations is the order of the highest derivative involved in any equation.
  - We are primarily interested in *first-order differential equations* of the form $\frac{dy}{dx} = f(x, y)$.
  - Differential equations of higher order may be reduced to systems of differential equations.

Example: free fall

- According the Newton's Law of Gravity, when an object is allowed to fall freely to the ground, its acceleration toward the ground is constant.
- We denote the height of the object at any time t by $h(t)$.
- The *velocity* (speed *and direction*) of the object is the *first derivative* of height, denoted $\frac{dh}{dt}$, $h'(t)$, or $\dot{h}(t)$.
- The *acceleration* of the object is the first derivative of velocity, or the second derivative of height, denoted $\frac{d^2h}{dt^2}$, $h''(t)$, or $\ddot{h}(t)$.
- Since it is constant, we have a *second-order* differential equation $h''(t) = g$.

The general solution for free fall

- We *solve* the differential equation by finding an equation with no derivatives in it.
- This is "simply" a process of integrating the equation as in basic calculus. Each integration lowers the order of the differential equation by one, and when the order reaches zero, we're done.

$$
\begin{aligned}
h''(t) &= g \\
\int h''(t)\,dt &= \int g\,dt \\
h'(t) &= gt + C_1 \\
\int h'(t)\,dt &= \int gt + C_1\,dt \\
h(t) &= \frac{1}{2}gt^2 + C_1 t + C_2
\end{aligned}
$$

- where C_1 and C_2 are *constants of integration*.

Specific solutions for free fall

- g, C_1, and C_2 are arbitrary constants, and we cannot use them to compute the height of the object numerically until we know their values.
- The values are determined from other facts about the problem. The "standard" problem specifies that
  - g is known from previous experimental measurements.
  - Time is measured in seconds since the object was set free.
  - The object was at rest at time 0, so $0 = h'(0) = g \cdot 0 + C_1 = C_1$.
  - The height of the object at time 0 was measured to be h_0, so $h_0 = h(t) = g \cdot 0^2 + C_2 = C_2$.

Example: Soap bubbles

- Why are soap bubbles spherical?
- The mathematical model of a bubble is based on a system of partial differential equations which characterize equality of air pressure inside and outside of the bubble.
- These differential equations have a *spatial parameter*, *i.e.*, the position of each point on the bubble. So differential equations need not be based on a time parameter (though in economic dynamics they are).
- The soap bubble model is completed by describing it as an optimization problem which minimizes surface area subject to the system of differential equations.

The exponential growth model

- The rate of increase of a population with more than sufficient food supply and no predators to avoid is proportional to the population:

$$\frac{dx}{dt} = ax$$

- Integrating gives

$$\begin{aligned} \frac{dx}{dt} &= ax \\ \frac{1}{x}dx &= a\,dt \\ \ln x &= at + C_1 \\ x &= e^{C_1}e^{at} = x_0e^{at} \end{aligned}$$

where $x_0 = e^{C_1}$ is interpreted as the population at time 0.

Logistic growth

- The rate of increase of a population with a restricted food supply and no predators to avoid is approximately proportional to the population for small populations, but decreases as population approaches carrying capacity. This can be interpreted as decreased fertility or as an increased death rate.

$$\frac{dx}{dt} = x\,(\beta - \delta x).$$

- Integrating.

  1. Separate the variables:
  $$\frac{1}{x\,(\beta - \delta x)}\,dx = dt.$$
  The fraction suggests integrating to a logarithm, but the argument will be a complicated function with a nontrivial derivative.

2. The form of the product in the denominator suggests a simplification (for calculus purposes) to a sum of simpler fractions:

$$\begin{aligned}
\frac{1}{x\,(\beta-\delta x)} &= \frac{\delta}{\beta}\left(\frac{\beta}{(\delta x)(\beta-\delta x)}\right) \\
&= \frac{\delta}{\beta}\left(\frac{(\delta x)+(\beta-\delta x)}{(\delta x)(\beta-\delta x)}\right) \\
&= \frac{\delta}{\beta}\left(\frac{1}{\beta-\delta x}+\frac{1}{\delta x}\right) \\
&= \frac{1}{\beta}\left(-\frac{-\delta}{\beta-\delta x}+\frac{1}{x}\right)
\end{aligned}$$

Why write the first term in parentheses as $-\frac{-\delta}{\beta-\delta x}$, not $-\frac{\delta}{\delta x-\beta}$? When integrating, the denominator becomes the argument to the logarithm function. A negative argument is nonsense, since we aren't using complex numbers. (Note: The formal integration and solution can be done if you ignore this fine point, but you get a different answer! It's wrong for the case where $x(0)$ is "small" and the population grows, but correct for the case where $x(0)$ is "too large," and the population shrinks.)

3. Substituting, rearranging, and integrating:

$$\int \frac{1}{x} - \frac{-\delta}{\beta - \delta x}\, dx = \int \beta\, dt,$$

gives

$$\ln x - \ln(\beta - \delta x) = \beta t + C.$$

4. Exponentiating and simplifying slightly:

$$\frac{x}{\beta - \delta x} = Ae^{\beta t},$$

where $A = e^C$ is derived from the integration constant.

5. Solving for x gives:

$$x = \frac{\beta A e^{\beta t}}{\delta A e^{\beta t} + 1}.$$

6. Setting $t = 0$ gives

$$x_0 \equiv x(0) = \frac{\beta A}{\delta(A+1)},$$

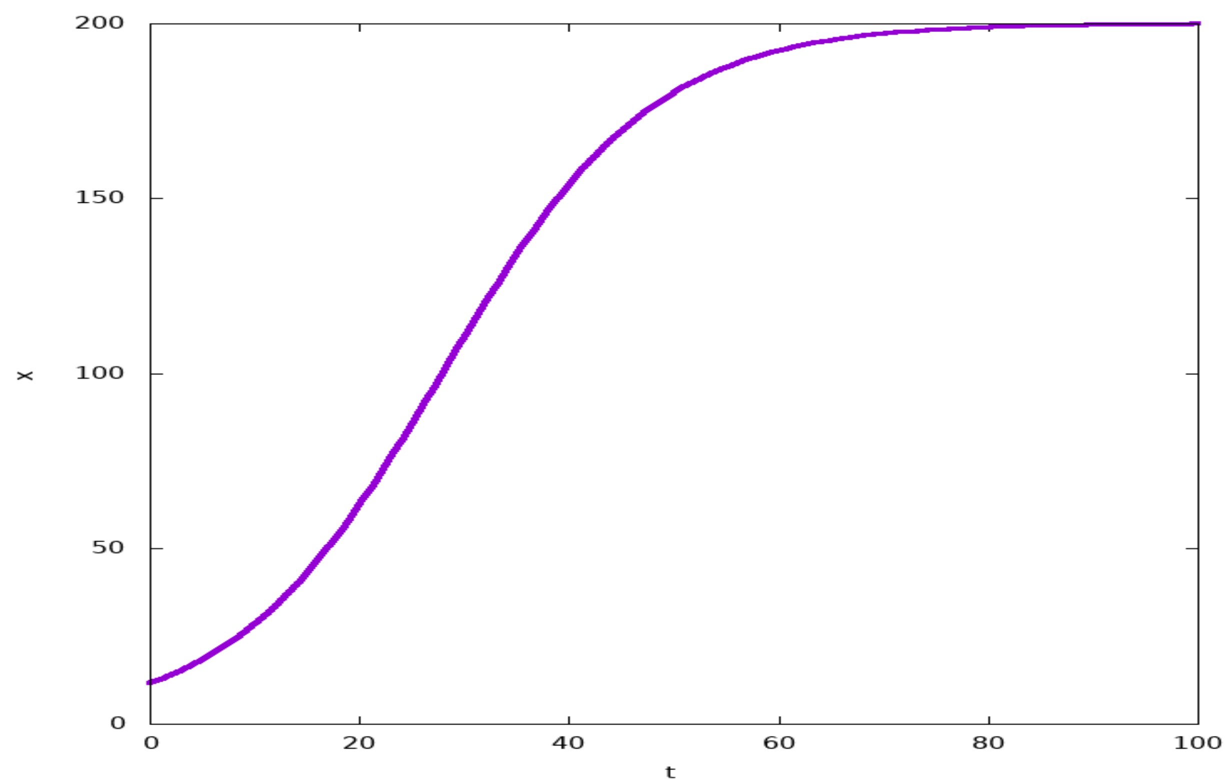
and solving for A gives

$$A = \frac{x_0}{\beta - \delta x_0}.$$

7. Substituting for A in the equation for x, multiplying numerator and denominator by $\delta x_0 - \beta$, and simplifying gives:

$$x = \frac{\beta e^{\beta t} x_0}{\beta - \delta x_0 + \delta x_0 e^{\beta t}}.$$

Logistic growth as time series.



Solving differential equations computationally

- In one sense, a differential equation always has a solution. That is, the fundamental theorem of calculus says that for an integrable function $f(t)$, $\int_{\ell}^{u} f(t)dt = F(u) - F(\ell)$, where $f = \frac{dF}{dt}$, and F is continuous and differentiable.

- In practice, we can always compute a time path for $f(t)$ by simulation (picking a value for $F(\ell)$, then setting $F(\ell + (n+1)\delta) = F(\ell + n\delta) + \delta f(\ell + n\delta))$ for δ "sufficiently small").

- However, this is not generally very useful in economic theory (though it is frequently used for examples and actual simulations).

Characterizing solutions

- Because of *resource limitations*, “explosive” growth by individual entities cannot continue indefinitely. From the point of view of individuals in an economy, there should be some stability.
- History shows that individuals *can* usefully predict (near) future conditions by assuming they won’t be (much) different from current conditions, so there is a degree of stability.
- For these reasons, not all differential equations are useful models of economic phenomena.
- We also want to be able to characterize the solutions in terms of
  - “where they settle down” (existence of steady states)
  - “how fast they settle down,” (stability of and speed of convergence to steady state), and
  - “optimal control” (where the direction and speed of change can be controlled by policy).

Approximating $y' = ax - by$ and $y(0) = a/b$

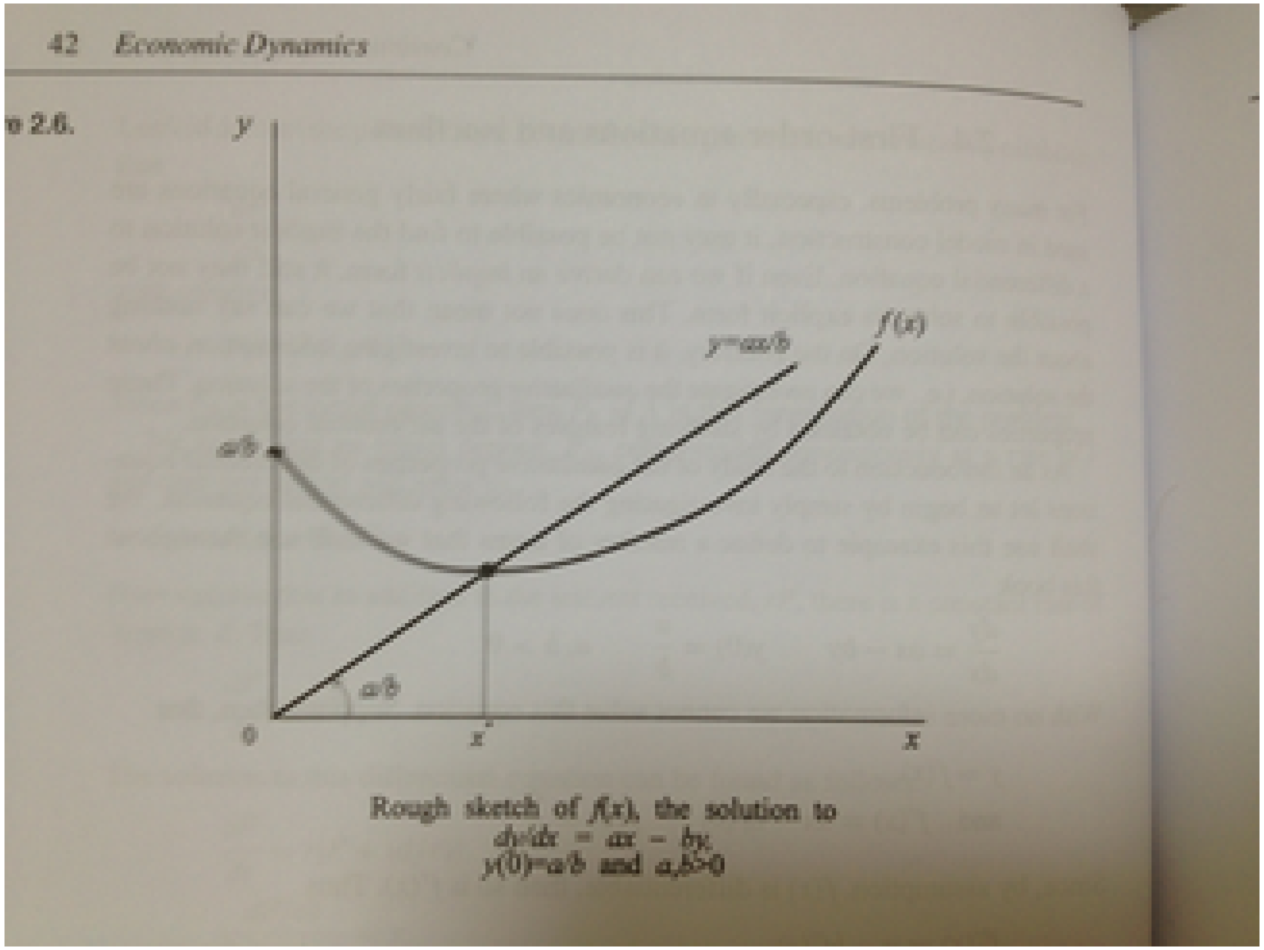
We can find an approximate specific solution to $y' = ax - by$ and $y(0) = a/b$ where $a > 0$ and $b > 0$ using the following procedure:

- Plot the point $(0, a/b)$ on the graph.
- Evaluate $\frac{dy}{dx}$ at $(0, a/b)$, getting $-a$, so the specific solution f (*i.e.*, $y = f(x)$ for all x) is downward-sloping at that point.
- Assume that f has a minimum. Then at that point $\frac{dy}{dx} = 0$, so $y = \frac{ax}{b}$. (We know the derivative exists because f is the solution to a differential equation!)
  - We need to check that the intersection of f and $y = \frac{ax}{b}$ is a minimum, not an inflection point, of f.

$$\frac{d^2y}{dx^2} = a - b\frac{dy}{dx} = a > 0,$$

    because $y = \frac{ax}{b}$ is defined so that $\frac{dy}{dx} = 0$, and f is convex, so this is a minimum.
- The curve must look something like the next graph.

Approximating $y' = ax - by$ and $y(0) = a/b$



Isoclines

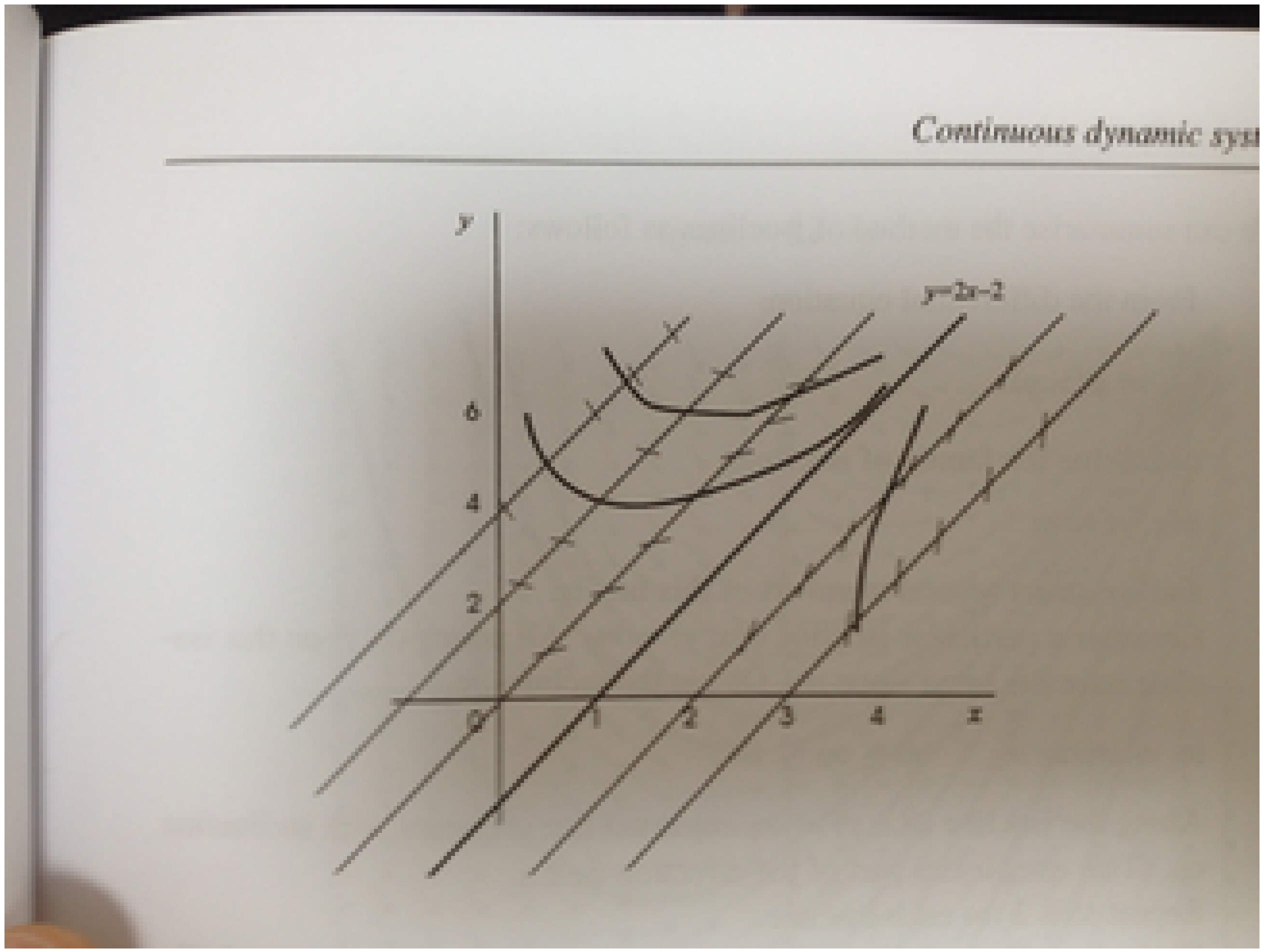
- "Cline" is a word (Greek or Latin) meaning slope. "Iso" means "equal" in the same language. Thus, "isocline" means "something" has the same slope in different places. In fact, an *isocline* is the set of all points where that "thing" has a given slope.

- In our specific solution, by definition, f has the same slope m for all points satisfying $m = \frac{dy}{dx} = ax - by$. So the *general* solution has isoclines with the parametric equation $y = \frac{ax-m}{b}$. m can be any number.

- Isoclines need not be linear.

- The isocline for $\frac{dy}{dx} = 0$ is special, because any steady state must be contained in that isocline.

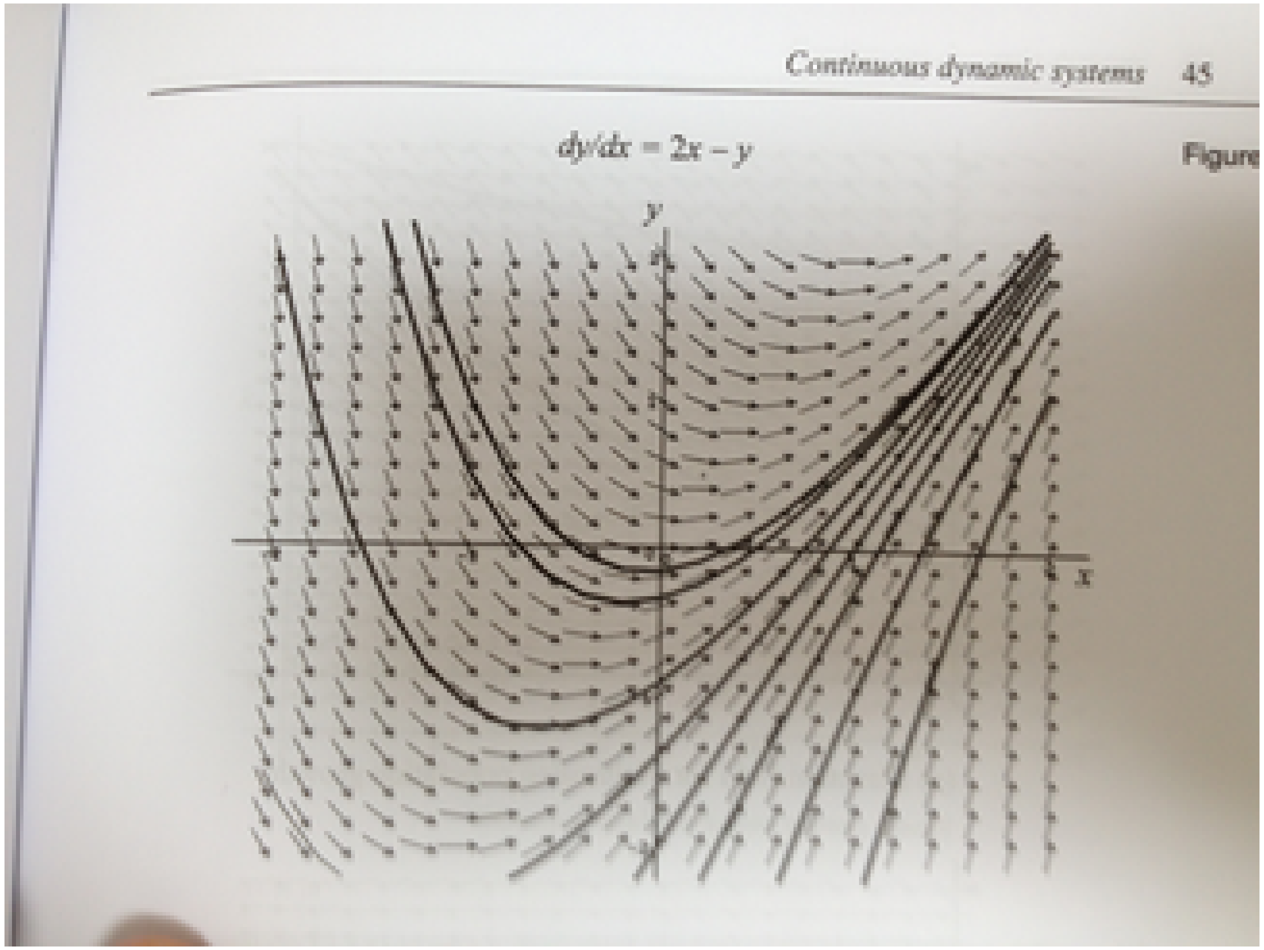
Isoclines of $y' = 2x - y$



Direction fields

- An alternative representation is to attach the slope implied by $\frac{dy}{dx}$ to each point.
- This is call a *direction field.*
- If we start at some point and "connect the arrows" head to tail, we get a curve called an *integral curve.* This is the graph of a specific solution.

Direction field of $y' = 2x - y$



Other properties

- In the image of the direction field, it seems that there's a "main stream" crossing from the 3rd quadrant through the 4th quadrant and then going up and to the right forever in the 1st quadrant.
- This is correct.
- Consider the isocline going through $(0, 0)$. The isocline has slope 2, but the direction field's value is 0. So the particular solution going through $(0, 0)$ will "move off" the isocline.
- Since our isoclines are linear with slope a/b, let's consider the condition $\frac{dy}{dx} = \frac{a}{b}$. Then $\frac{a}{b} = ax - by$, and $y = \frac{a}{b}x - \frac{a}{b^2}$. Thus this condition identifies an isocline!
- Of course this is harder to do for nonlinear $\frac{dy}{dx}$, but when possible it is very useful.

Homework Submissions

1. Submit your homework *by email to*

   `"Economic Dynamics" <dynamics-hw@turnbull.sk.tsukuba.ac.jp>`

   (Note: `dynamics-hw@turnbull...`, not `turnbull@sk....`) The `Subject:` should be `FH27041 Homework #1`. (For assignments #2, #3, and so on, adjust the homework number.)

2. Without the class number and the homework assignment in hankaku romaji, your email may get lost. Use the class number above, even if you are registered according to a different code.

3. Your email must contain your *name* and *student ID number*.

4. You should receive an automatic acknowledgement of your homework by email. Please keep this message in case I lose your homework. (It hasn't happened yet as far as I know, but hardware can fail, *etc.*)

Homework Format

5. For simple answers, I *strongly* prefer *plain text* or TEX to *Word documents* and *HTML*. In plain text, you may write subscripts using programming notation (*i.e.*, X_t becomes `X[t]`), and superscripts using the caret (*i.e.*, X^t becomes `X^t`) or double-star (X^t becomes `X**t`).

Homework 1, due 2018-10-12, 11:00am

In class we discussed the case of an object in free fall, described by the equation $\ddot{h}(t) = g$, where t is time in seconds and h is the height above the ground.

In all of the problems below, use $g = -9.80\,\mathrm{m/sec}^2$ (m is meters and sec is seconds).

1. Give the specific solution for the "typical exam problem," where the object is dropped (*i.e.*, starts at rest, or velocity zero) from a height of 1000 m.

2. Give the specific solution assuming the object is shot straight up from a cannon on the ground (height 0) at time 0 with velocity 100 m/sec.

3. Give the specific solution assuming the object is a rocket fired straight up that runs out of fuel after 30 seconds at a height of 1000 m with a velocity of 100 m/sec.

You only need to give information about solutions starting from the initial condition given until the time the object hits the ground.

Homework 2, due 2018-10-12, 11:00am

1. Suppose that members of a population are born at the birth rate β and die from natural causes at the death rate δ (which do not depend on population P, that's what "natural causes" means here). Derive the equation for rate of population increase, and compare it to the "reduced form" model $\dot{P} = \alpha P$.

2. Verify that the logistic growth equation $P(t) = \dfrac{\beta}{\gamma + [\frac{\beta}{P(0)} - \gamma]e^{-\beta t}}$ is a solution to the logistic differential equation $\dfrac{dP}{dt} = P(\beta - \gamma P)$ by differentiating the logistic growth equation, and showing that the result is the differential equation. (This equation is equivalent to the one derived in class.)

3. Explain how the interpretation of γ in the constrained growth model (*i.e.*, the logistic growth model) differs from that of δ in the case without resource constraints described in Problem 1. What (possibly variable) expression in this model corresponds to δ in the unconstrained case?

Writing homework answers

This page is under development. More information will be added as I notice issues.

Many of the tasks assigned in homework are expressed using idioms specific to this class.

solve Also **give a solution** or **derive**. You *must* show your work. Obvious calculations of common operations, such as the $6 \times 5 \times 4 \times 3 \times 2 \times 1$ in $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ may be omitted, but even slightly more complex operations such as ${}_6C_3 = \dfrac{6!}{3!3!} = 20$ should be written out.

discuss Most important, relate the computation to the real problem in economics (or physics or biology for some of the "toy" examples). Especially mention anything paradoxical, surprising, or extreme about the interpretation of the result in context of the real problem.

compare Like **discuss**, but more specific: you should use statements of the form "*this* is the same as *that*," "*this* is different from *that*," and (best) "*this* is similar to *that*, except ...."

show ... is Often you need to transform one of the expressions to the other. You must show your work, not just "$expr_1 = expr_2$ (same!)"

notation You may define your own notation whenever convenient. For example, in HW#2, Q#3 you're asked to compare δ in Q#1 to δ in Q#2. This gets confusing and long winded (*i.e.*, because you write "δ of Problem 1" over and over again). It may be useful to rewrite one of the results (in this case, that of Q#2) by substituting γ for δ everywhere.