Introducing Overlapping Generations Models and Chaotic Dynamics

Lecture 6 of Economic Dynamics

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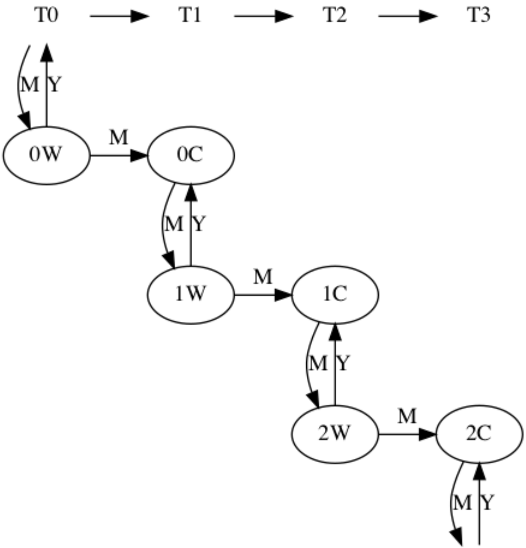
Overlapping generations models, and an introduction to chaotic dynamics.

Overlapping generations models

- Up to the present, we've considered dynamic constraints on single homogeneous entities. Examples:
  - In Solow's model, the single "interesting" entity is the *representative worker/consumer*, which we derive using the special properties of CRTS production.
  - In the fishery, the "interesting" entity is the *population* of fish (or whales). Although the fisherman do interact in equilibrium, the dynamic constraint is on the population.
- By contrast, in an *overlapping generations* (OLG) *model*, there are constraints between agents existing at the same time and a given agent across time periods.

OLG constraints



A simple OLG model

Follows Ch. 17 of Lucas and Stokey.

- The economy has a constant population of agents (worker/ consumers).
- The agent lives for two periods, working when young and consuming when old. (This is a *technical* assumption, convenient in notation, computation, and interpretation because the number of workers equals the number of consumers equals half the population.)
- The utility function is $U(c, l) = -H(l) + V(c)$.
- There is a single, non-storable good, produced with a linear technology $y = xl$, where X is generated by a *Markov process*. (This means that x_{t+1} is generated by a random variable which may depend on x_t but nothing else.)
- There is a constant supply of *fiat money* (government-issued, as with yen and dollars) M.

How the OLG model works

- We make the *technical* assumption that there's one person in each generation. (Like Solow's model, this one is CRTS.)
- Based on an assumption of equilibrium, markets will clear:
  - The young worker will supply labor l, produce $y = xl$, and receive all the money M from the old consumer.
  - The old consumer will consume $c = y$, and pay all the money M to the young worker.
- The old consumer's behavior is forced: they have money, they buy the good in a competitive market, so they'll spend all the money and buy all the good.

The worker's model

- When young, the worker dislikes working, with the usual "decreasing returns to scale" conditions: $H : [0, L) \to \mathbf{R}_+$ satisfies $H'(l) > 0$ and $H''(l) > 0$ for all l, and $H'(0) = 0$ and $\lim_{l \to L} H'(l) = \infty$ (Inada!).
- When old, the consumer likes consuming, with decreasing marginal utility. $V : \mathbf{R}_+ \to \mathbf{R}_+$ satisfies $V'(c) > 0$ and $V''(c) < 0$.
- The equilibrium is characterized by
  - the "price" (of money in goods, not the reverse!) $p(x)$, which depends on the state of the world (random worker productivity),
  - the "labor supply" function $n(x)$ (n depends on x, not the wage), and
  - market-clearing $xn(x) = M/p(x)$.
- When old, the worker born at t consumes $x_t n(x_t)(p(x_t)/p(x_{t+1}))$.

The worker's optimization

- The worker chooses $l = n(x)$ to maximize
$$-H(l) + \mathcal{E}_\xi \left[V(xl\frac{p(x)}{p(\xi)}) \mid x \right]$$
where the worker knows her own productivity x (invert the price function p) but productivity of later generations is random ξ.
- Given a price function p, the first-order condition for n solves
$$H'(n(x)) = \mathcal{E}_\xi \left[V'(xn(x)\frac{p(x)}{p(\xi)}) \mid x \right]$$
(there are no n' because x is a parameter known to the worker, not a choice variable—the worker chooses a different n for each x).
- Substituting from the market-clearing conditions for this period and next gives
$$n(x)H'(n(x)) = \mathcal{E}_\xi \left[\xi n(\xi) V'(\xi n(\xi)) \mid x \right]$$

The equilibrium

- Suppose x has a distribution independent of time and across time. Then $n(x) = \bar{n} > 0$ for all x.
- Under certain conditions on the Markov process, and the same assumptions on production and utility, for a general process (*i.e.*, serially correlated x), there exists

$$f^*(x) = \mathcal{E}_\xi \left[\phi(\xi\zeta^{-1}(f^*(\xi))) \mid x \right]$$

where $\phi(y) = yV'(y)$ and $\zeta(l) = lH'(l)$.
- **Note 1:** f^* is defined as a fixed point, like a value function.
- **Note 2:** This is not a differential equation model. $p(x), n(x)$ are determined "independently" (in a sense) from $p(x'), n(x')$ for $x \neq x'$.

The Lucas "Islands" model

We change the preceding model in the following way.

- We have a *deterministic* production function, $y = l$ (*i.e.*, $x \equiv 1$).
- Workers (young) are randomly assigned to two "islands". $\frac{\theta}{2}$ go to one island and $1 - \frac{\theta}{2}$ to the other, where $0 < \underline{\theta} < \bar{\theta} < 2$.
- Consumers (old) are assigned randomly to the two islands such that each island has half the old population and half the money.
- The government pays interest on or taxes the money stock randomly, such that $m_{t+1} = x m_t$ for a worker who received m_t, and x is random between $0 < \underline{x} < \bar{x} < \infty$.
- The varying ratio of workers to consumers is a *real* shock (affects available consumption per old person, while the monetary shock is *nominal*. *Nominal* means there is no change to physical possibilities. It does reduce the accuracy of workers' assessment of their future consumption, and thus their incentive to work.

Market conditions

- Here the *state* of the economy is 2-dimensional: (x, θ). (x is the increase factor for the money supply, θ the population assignment between islands.)
- As before (equilibrium) price of consumption and (optimal) labor "supply" are functions of the state: $p(x, \theta)$ and $n(x, \theta)$.
- In the previous model, $pc = \frac{M}{I}$, where the latter is constant, so we can invert the equilibrium price function $x = p^{-1}(p)$, and it doesn't matter if the workers can observe x, by assumption of competition they know p and can deduce x from that. Here, M is *uncertain*, so in equilibrium, by observing p and given x you can figure out θ, and *vice versa*. But you can't deduce both.
- Assume x and θ independent for each t, and (x, θ) *i.i.d.* over time.

Equilibrium conditions

- Labor supply $l = n(x, \theta)$ maximizes over l

$$-H(l) + \mathcal{E}_{\bar{x},\bar{\theta}} \left\{ \mathcal{E}_{x',\theta'} \left\{ V \left[\frac{x' l p(\bar{x}, \bar{\theta})}{\bar{x} p(x', \theta')} \right] \mid p(\bar{x}, \bar{\theta}) = p(x, \theta) \right\} \right\}$$

  where
  - x, θ are current values, unknown to the worker
  - The worker does know p, so can deduce that all $\bar{x}, \bar{\theta}$ such that $p(\bar{x}, \bar{\theta}) = p(x, \theta)$, and so take expectation over only those values of $\bar{x}, \bar{\theta}$, and
  - given $\bar{x}, \bar{\theta}$, the worker can take the expectation of the consumption next period based on the independent distribution of x', θ', the values of the nominal and real shocks respectively.
- Market clearing: for all (x, θ)

$$n(x, \theta) p(x, \theta) = \frac{x}{\theta}$$

Interpretation

- The utility function looks very complicated because of all the bars and primes, but the important aspects are
  - the description of each variable on the previous slide, and
  - most important, the conditioning equation $p(\bar{x}, \bar{\theta}) = p(x, \theta)$, which shows how the nominal shock and the real shock are confounded (confused) by a rational consumer/worker.
- The end result is that we can show that $\frac{dn}{dp} > 0$, which is a Philips curve, *i.e.*, a positive relationship between employment and inflation.

- The dynamic models so far are simple.
- In the Solow model with the standard conditions on the production function, for example, there are four regions in $[0, \infty)$:
    - the singleton k $= 0$, the unstable steady state
    - the region $0 < k < k^*$, where $k(t)$ is monotonically increasing in t
    - the singleton $k = k^*$, the stable steady state, and
    - the region $k > k^*$, where $k(t)$ is monotonically decreasing in t.
- The renewable resource model is only a little more complicated.
    - Possibly an additional region from 0 to the unstable steady state where $Z(t)$ is monotonically decreasing in t.

A dynamic system: The Hénon map

- A discrete time iterated function in two dimensions:

$$\begin{aligned} x_{t+1} &= 1 - ax_t^2 + y_t \\ y_{t+1} &= bx \end{aligned}$$

```
import numpy as np
def make_henon(x, y, a=1.4, b=0.3):
    while True:
        yield x, y
        x, y = 1 - a*x**2 + y, b*x
def classic_henon(n):
    henon = make_henon(x=0.3, y=0.3)
    xys = np.empty((2,n))
    for i, (x, y) in enumerate(henon()):
        if i >= n: break
        xys[0][i], xys[1][i] = x, y
    return xys
xys = classic_henon(100000)
```

A Hénon sequence

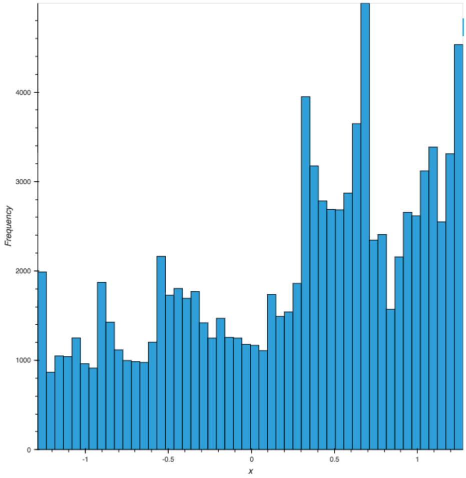
- Let's look at the first 10 points of the sequence:

```
>>> list(zip(xys[0][:10], xys[1][:10]))
[(0.3, 0.3),
 (1.174, 0.09),
 (-0.839586399999, 0.352199999999),
 (0.36533254770, -0.251875919999),
 (0.561269061418, 0.109599764312),
 (0.668567621285, 0.168380718425),
 (0.542604988501, 0.200570286385),
 (0.788382043419, 0.162781496550),
 (0.292616751609, 0.236514613025),
 (1.11664022437, 0.0877850254827)]
```

- No obvious pattern here.

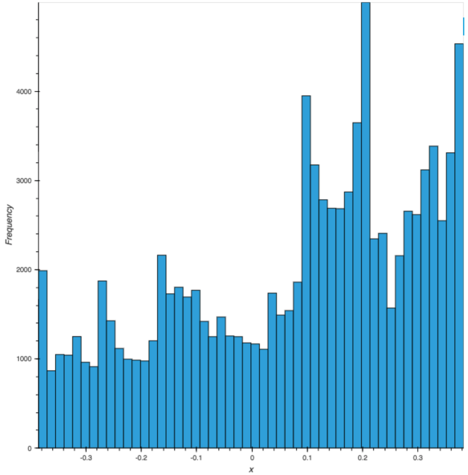
The histogram of x

Ranges from -1.3 to +1.3, with frequencies up to 5000.



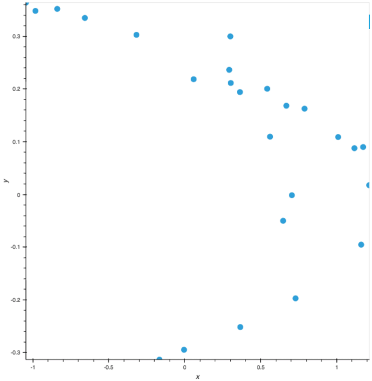
The histogram of y

Ranges from -0.4 to +0.4 with frequencies up to 5000.

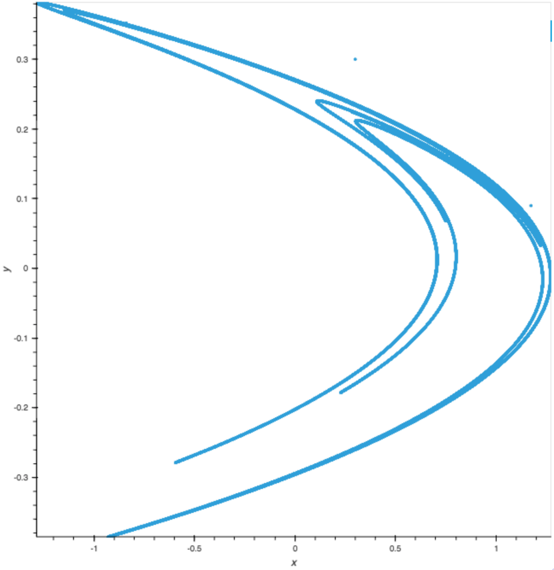


Two-dimensional distribution

- There's no obvious pattern in the (short) data series or the histograms. Random? (but not uniform).
- Graph a small sample of points.
- Clearly not very random.



The Hénon attractor phase diagram



The Hénon attractor

- There is fine structure in the phase diagram, which can be seen by generating enough points and zooming in on smaller and smaller regions. What looks like a thick curve is actually composed of several parallel curves, and each of those has a similar structure, "all the way down."
- The structure is *fractal*, which means that even though the graph looks like a curve that is smooth and continuous with respect to time, in fact the next point jumps around with each iteration in a chaotic way.
- It's called "fractal" because there is a sense in which the attracting region has more than 1 dimension even though it has zero area. By some definitions, this dimension is fractional (like 3/2).

The logistic map

- The Hénon map is two-dimensional, and quite complicated.
- To understand chaos, we simplify to a one-dimensional system.
- Complex dynamics can exist in a one-dimensional discrete time system (impossible with a differential equation), *e.g.*, by iterating the logistic map $f(x) = ax(1 - x)$.
- Any concave, single-peaked map gives the same result (it doesn't even need to be differentiable).

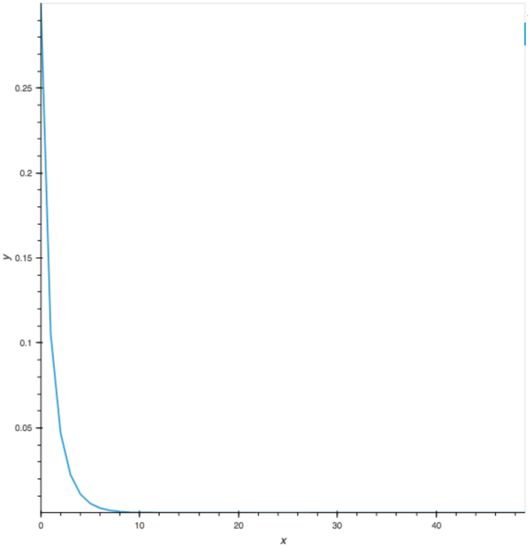
Iterating the logistic map

- We consider "time series" for various values of $a \in [0, 4]$.

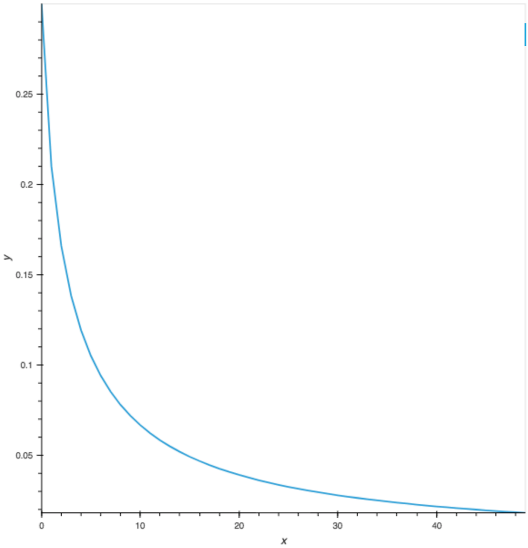
```
def bell(a, x):
    while True:
        yield x
        x = a*x*(1-x)

def bell_series(a, x=0.3, n=50):
    xs = []
    b = bell(a, x)
    for i in range(n):
        xs.append((i, next(b)))
    return xs
```

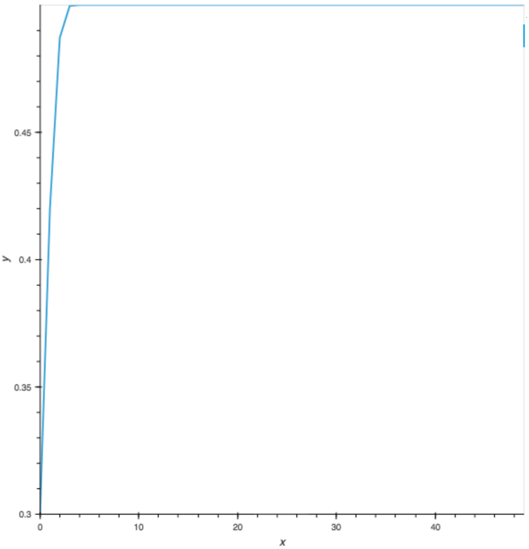
Logistic map: $a = 0.5$



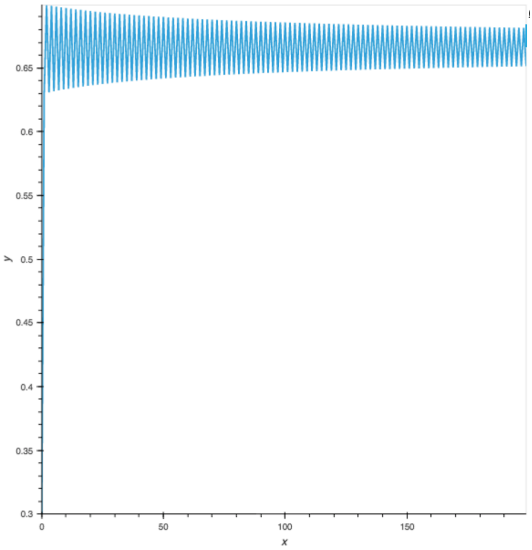
Logistic map: $a = 1.0$



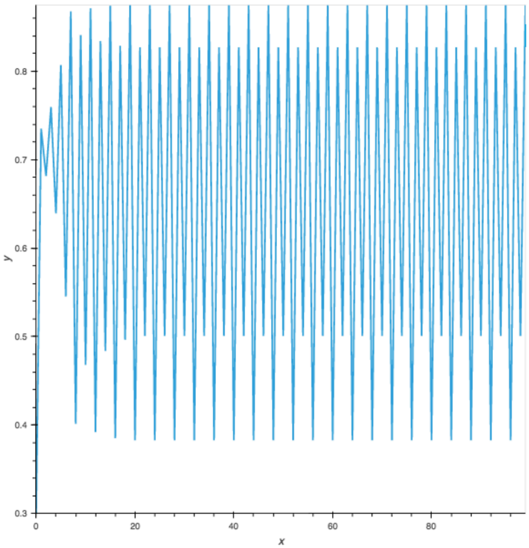
Logistic map: $a = 2.0$



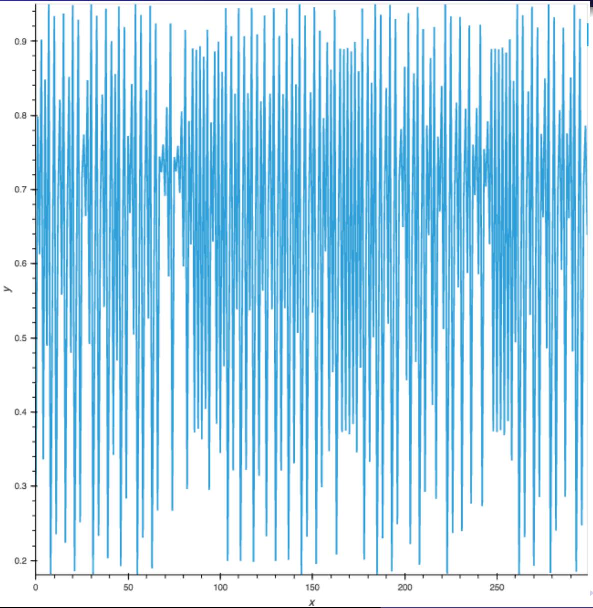
Logistic map: $a = 3.0$



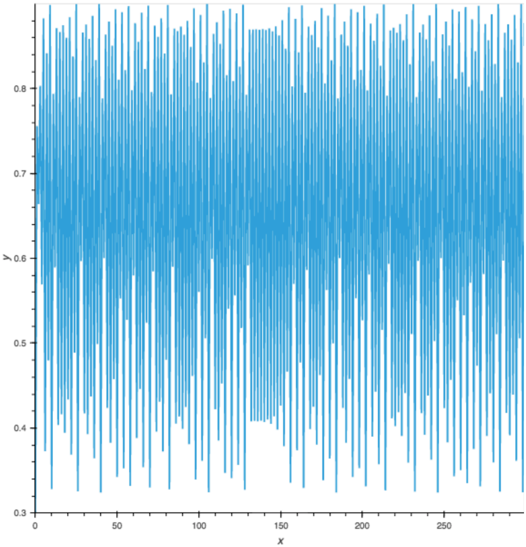
Logistic map: $a = 3.5$



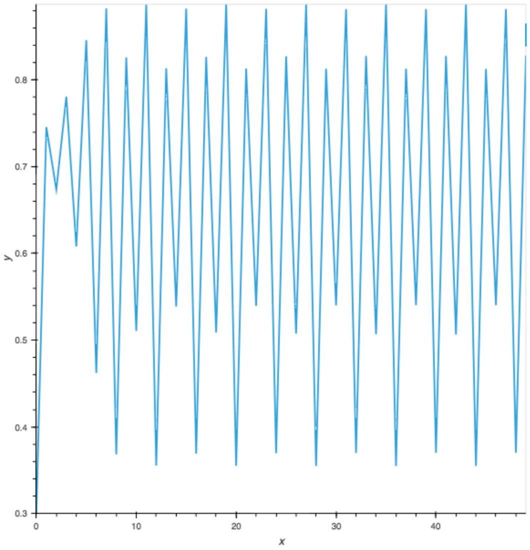
Logistic map: $a = 3.8$



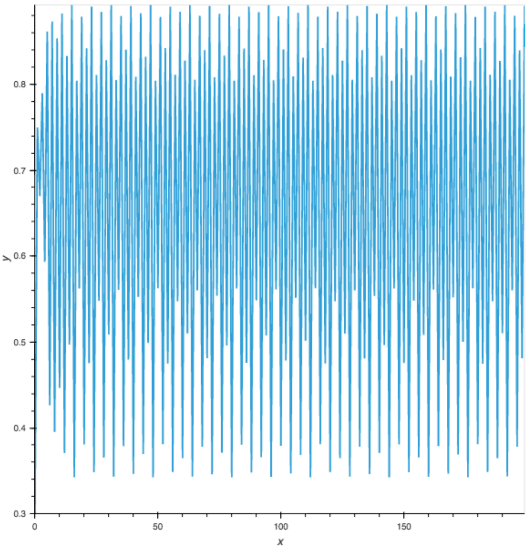
Logistic map: $a = 3.6$



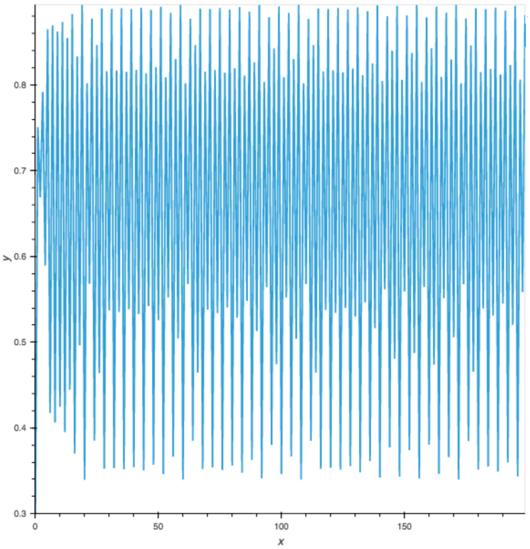
Logistic map: $a = 3.55$



Logistic map: $a = 3.57$



Logistic map: $a = 3.575$



- The *Sharkovsky Theorem* says that for an iterated function f

$$\begin{array}{ccccccccc} & & 2^0 \cdot 3 & \Longrightarrow & 2^0 \cdot 5 & \Longrightarrow & 2^0 \cdot 7 & \Longrightarrow & \dots \\ & \Longrightarrow & 2^1 \cdot 3 & \Longrightarrow & 2^1 \cdot 5 & \Longrightarrow & 2^1 \cdot 7 & \Longrightarrow & \dots \\ & \Longrightarrow & 2^2 \cdot 3 & \Longrightarrow & 2^2 \cdot 5 & \Longrightarrow & 2^2 \cdot 7 & \Longrightarrow & \dots \\ \dots & \Longrightarrow & 2^3 & \Longrightarrow & 2^2 & \Longrightarrow & 2^1 & \Longrightarrow & 2^0 \end{array}$$

  where "n" means f has a period-n cycle for some x_0.
- A cycle of period $2^2 = 4$ implies there is a cycle of period $2^1 = 2$ and one of period $2^0 = 1$ (*i.e.*, a fixed point $f(x) = x$).
- The *Li-Yorke* theorem says that period three implies that almost all x_0 in the domain of f) are *asymptotically aperiodic*: there is no cycle, and there is $\epsilon > 0$ such that for given t and $s > 0$, no matter how small $|x_t - x_{t-s}|$ is, there is a time $T > t$ such that $|x_T - x_{T-s}| > \epsilon$.

Chaos in economics

- Chaos has been proposed as a source of volatility.
- How can economists distinguish chaotic dynamics from stochastic dynamics?
- In the Hénon map, the values jump around, but accumulate in an attractor.
- Independence of x and y implies points should be uniformly distributed in two dimensions. But they are not.
- Chaos would arise because of *autoregression*, needing no disturbance. In stochastic models we usually have reason to believe that disturbances in different equations are independent.
- Independence implies random distribution in all dimensions. In a chaotic model, there will be regions that the disturbances “avoid”, as in the right center of the Hénon map.